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MULTIPLIERS OF SEGAL ALGEBRAS AND RELATED CLASSES ON THE REAL LINE

S. POORNIMA

Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012

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For any Segal algebra S on the real line R we define a new Segal algebra $D(S)$ consisting of functions of S whose first order distributional derivative belongs to S. We show that the space of multipliers on $D(S)$ is the same as that on S. (Theorem 2.4). Applying this result we obtain the multipliers of the Segal algebra $L^A(R)$ immediately. As a more interesting application of this result we obtain an infinite family of Segal algebras on R whose multiplier spaces are the same as that of the Wiener algebra W (Proposition 2.6). We also compute by fairly simple arguments several other multiplier spaces of function spaces on the real line (see, for example, Propositions 3.8, 3.10, 4.3).

INTRODUCTION

This paper deals with the multipliers of Segal algebras on the real line R and as well with those of some related classes of functions. For every Segal algebra S on R we define a new Segal algebra $D(S)$ consisting of functions of S whose first order distributional derivative belongs to S. We show that (Theorem 2.4) the space of multipliers on $D(S)$ is the same as that on S. As an application of this result we get an infinite family of Segal algebras on R whose multiplier spaces are the same as that of the Wiener algebra W.

Krogstad (1976) has remarked that the Wiener algebra (and its generalizations on the non compact Abelian group) seems to be the only known Segal algebra whose space of multipliers strictly contains the bounded measures. Our result shows that, in fact, there is an infinite class of Segal algebras distinct from W, with this property.

We also compute by fairly simple arguments several other multiplier spaces of function spaces on the real line (see, for example, Propositions 3.8, 3.10). In particular, we consider the multipliers of the space V, introduced by Burnham and Goldberg (1975). V is actually the analogue of W obtained by replacing continuous functions by bounded functions. We establish the equivalence of several multiplier spaces for V and W.

Finally, using a technique of Hormander (1960) we prove that the multipliers from $L^1 \cap C_0$ or $L^1 \cap L^p$ into the Wiener algebra W are given by functions of V.

1. Preliminaries

Let R denote the additive group of real numbers. $L^1(R)$ is the Lebesgue space of measurable functions f on R which are absolutely integrable with the norm given by

$$\| f \|_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} | f(t) | \, dt.$$

$C_0(R)$ is the space of all continuous functions on R which vanish at infinity. $M(R)$ is the convolution algebra of bounded regular Borel measures on R and $P(R)$ is the space of pseudomeasures on R. BV will denote the set of functions of which are of bounded variation on R and normalized by

$$g(-\infty) = g(-\infty + 0) = 0, \; g(x) = \tfrac{1}{2} \{g(x - 0) + g(x + 0)\}.$$

The Fourier transform of $f \in L^1(R)$ is given by

$$\hat{f}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-iux) f(x) \, dx$$

for any $u \in R$. The set $B(R)$ consisting of those functions in $L^1(R)$ whose Fourier transforms have compact support is a dense subspace of $L^1(R)$. If $x \in R$, the translation operator τ_x is defined by

$$\tau_x f(y) = f(y - x), \quad y \in R.$$

By $f * A$ we mean $\{f * g : g \in A\}$ where $*$ denotes the convolution operation.

A linear subspace $S(R)$ of $L^1(R)$ is called a Segal algebra if the following four conditions are satisfied :

(i) $S(R)$ is dense in $L^1(R)$.

(ii) $S(R)$ is a Banach space under a norm $\| \; \|_S$ and $\| f \|_S \geqslant \| f \|_1$ for all $f \in S(R)$.

(iii) If $f \in S(R)$ and $y \in R$, then $\tau_y f \in S(R)$ and the mapping $y \to \tau_y f$ is continuous from R into $S(R)$.

(iv) $\| \tau_y f \|_S = \| f \|_S$ for all $f \in S(R)$ and all $y \in R$.

For various properties of Segal algebras see Reiter (1968, 1971).

The following Segal algebras are of interest to us :

1. The space $W(R)$ of continuous functions f satisfying

$$\sum_{K=-\infty}^{\infty} \max_{x \in [K, K+1]} | f(x) | < \infty$$

endowed with the norm

$$\| f \|_W = \sup_{m \in R} \sum_{K=-\infty}^{\infty} \max_{x \in [K, K+1]} | f(x + m) |$$

is a Segal algebra. This is evidently the well known Wiener (1932) algebra.

2. The space $L^A(R)$ of absolutely continuous functions f in $L^1(R)$ with the derivative f' in $L^1(R)$ is a Segal algebra if we set

$$\| f \|_{L^A} = \| f \|_1 + \| f' \|_1$$

as the norm.

3. The space $L^1 \cap C_0(R)$ is a Segal algebra under the norm

$$\| f \|_{L^1 \cap C_0} = \| f \|_1 + \| f \|_\infty, \; f \in L^1 \cap C_0(R).$$

In addition, we shall consider the following classes of functions on R.

4. All functions in $L^1(R)$ which are bounded on R. This is $L^1 \cap L^\infty(R)$.

5. All bounded functions in $L^1(R)$ which are integrable in the sense of Riemann in every finite interval. This we denote by $L^1 \cap RI$.

6. All functions in $L^1(R)$ which are of bounded variation on R. This is $L^1 \cap BV$.

7. All functions f bounded on R such that

$$\sum_{K=-\infty}^{\infty} \operatorname*{ess\,sup}_{x \in [K, K+1]} | f(x) | < \infty.$$

This class, denoted by V, is introduced by Burnham and Goldberg (1975). It can be shown that V is a Banach space under the norm

$$\| f \|_V = \sup_{m \in R} \sum_{K=-\infty}^{\infty} \operatorname*{ess\,sup}_{x \in [K, K+1]} | f(x + m) | , \; f \in V.$$

Also, we note that $W(R)$ is nothing but the class of continuous functions in V.

Among the classes defined above the following inclusion relations are valid :

$$L^1 \cap C_0 \subset L^1 \cap RI \subset L^1 \cap L^\infty \subset L^1 \quad \ldots(1)$$

$$L^A \subset L^1 \cap BV \subset L^1 \cap RI \quad \ldots(2)$$

$$L^A \subset W \subset L^1 \cap C_0 \quad \ldots(3)$$

$$W \subset V \subset L^1 \cap L^\infty \quad \ldots(4)$$

$$L^1 \cap BV \subset V. \quad \ldots(5)$$

The inclusions (1), (2) and (4) follow from the definitions. The inclusion in (3) is proved by Wang (1972). The only nontrivial inclusion is (5) which is proved independently below though it is implied in the result on multipliers given by Burnham and Goldberg (1975).

We shall now verify that $L^1 \cap BV \subset V$. Let $f \in L^1 \cap BV$ and let V_f denote the total variation of f. Given $\epsilon > 0$, to each integer K, we set

$$\epsilon_K = \frac{\epsilon}{2^K} \quad \text{and} \quad m_K = \operatorname*{ess\,inf}_{x \in [K, K+1]} | f(x) | .$$

There exists $x_0 \in [K, K+1]$ such that

$$| f(x_0) | < m_K + \epsilon_K$$

and for any $x \in [K, K+1]$ we have

$$| f(x) | < | f(x) - f(x_0) | + | f(x_0) | < V_f [K, K+1] + m_K + \epsilon_K$$

$$< V_f [K, K+1] + \int_K^{K+1} | f(y) | \, dy + \epsilon_K$$

where $V_f [K, K+1]$ denotes the variation of f over $[K, K+1]$ so that

$$\operatorname*{ess\,sup}_{x \in [K, K+1]} | f(x) | < V_f [K, K+1] + \int_K^{K+1} | f(y) | \, dy + \epsilon_K . \qquad \ldots(6)$$

Since ϵ is arbitrary, summing (6) with respect to K, we get

$$\sum_{K=-\infty}^{\infty} \operatorname*{ess\,sup}_{x \in [K, K+1]} | f(x) | < \| f \|_1 + V_f < \infty,$$

and thus $f \in V$. This proves our asserted inclusion.

Multiplier Notation

If A and B are subsets of tempered distributions on R, then (A, B) will denote the class of functions φ defined on R such that $\varphi \hat{A} \subset \hat{B}$. $\hat{A}$ stands for the set of Fourier transforms of distributions in A.

If A and B are translation invariant Banach spaces of functions on R, then $M(A, B)$ will stand for the set of all bounded linear transformations from A to B which commute with translations.

By a multiplier, we mean an element of either (A, B) or $M(A, B)$.

2. The Segal Algebra $D(S)$

We shall now introduce a Segal algebra $D(S)$ for any given Segal Algebra S on R and prove that the space of multipliers $M(S, S)$ is the same as the space of multipliers

$M(D(S), D(S))$. As a consequence we characterize the multipliers of the Segal algebra $L^A(R)$. Also, this enables us to produce an infinite class of Segal algebras on R having unbounded measures as multipliers (Proposition 2.6) in contrast to the statement of Krogstad (1976). (See Remark 2.7) below).

Theorem 2.1 — Let S be any Segal algebra on R. Define

$$D(S) = \left\{ f \in S : \frac{df}{dx} \in S \right\}$$

where $\frac{df}{dx}$ is the first order distributional derivative of f. Endowed with the norm

$$\| f \|_{D(S)} = \| f \|_S + \left\| \frac{df}{dx} \right\|_S, \; f \in D(S),$$

$D(S)$ is a Segal algebra on R.

PROOF: We shall first show that $D(S)$ is complete under its norm. If $\{f_n\}$ is a cauchy sequence in $D(S)$, by the completeness of S, there exist $f, g \in S$ such that

$$\| f_n - f \|_S \to 0 \text{ and } \left\| \frac{df_n}{dx} - g \right\|_S \to 0 \text{ as } n \to \infty.$$

This implies that $\hat{f}_n(u) \to \hat{f}(u)$ uniformly and $iu\hat{f}_n(u) = \frac{d\hat{f}}{dx}(u) \to \hat{g}(u)$ uniformly. This would mean $iu\hat{f}(u) = \hat{g}(u)$ for all $u \in R$. Hence, by the uniqueness of Fourier transforms,

$$\frac{df}{dx} = g$$

as tempered distributions. Since $g \in S$, $\frac{df}{dx} \in S$ and thus $D(S)$ is complete.

The translation invariance and the continuity of translation property for $D(S)$ can be established using the corresponding properties of the Segal algebra S.

Now to prove that $D(S)$ is dense in $L^1(R)$ we shall show that the space

$$B(\mathrm{R}) = \{ f \in L^1(R) : \hat{f} \text{ has compact support} \}$$

is contained in $D(S)$. $B(R)$ is contained in every Segal algebra (Reiter 1971) and hence in particular in L^A and S. Let $f \in B(R)$. Then $\frac{df}{dx} \in L^1(R)$ since $B(R) \subset L^A$. Also $\frac{d\hat{f}}{dx}(u) = iu\hat{f}(u)$ implies that $\frac{d\hat{f}}{dx}$ has compact support. Hence $\frac{df}{dx} \in B(R)$. Thus $\frac{df}{dx} \in S$ from which it follows that $B(R) \subset D(S)$. Hence $D(S)$ is a Segal algebra.

Remark 2.2 : Observe that functions in $D(S)$ are in fact absolutely continuous [see Schwartz (1966)]. $D(S)$ can also be defined as $\{f \in S : f$ absolutely continuous and $f' \in S\}$. So we see that $D(L^1)$ is the Segal algebra $L^A(R)$.

Theorem 2.3 — Let S be Segal algebra on R. Then the Segal algebra $D(S)$ is topologically isomorphic to S, by an isomophism which commutes with translations.

PROOF : Consider the function $k(x) = \begin{cases} e^{-x}, & x \geqslant 0 \\ 0, & x < 0. \end{cases}$

Clearly $k \in L^1(R)$. Also, k is a fundamental solution of the differential operator $\left(\frac{d}{dx} + 1\right)$.

Define a map $K : f \to k * f$ on S. Then K is a one to one linear map from S into $D(S)$. For, if $f \in S$, then as S is an ideal in L^1, clearly $k * f \in S$. To show that $k * f$ is actually in $D(S)$ observe that for every test function ψ we have

$$\left\langle \psi, \frac{d}{dx}(k * f) \right\rangle = \left\langle \psi, \frac{dk}{dx} * f \right\rangle$$

by Fubini's theorem. Hence we can write

$$\frac{d}{dx}(k * f) + k * f = \left(\frac{d}{dx} + 1\right) k * f = \delta * f = f$$

where δ is the Dirac delta measure. Now $\frac{d}{dx}(k * f) = f - k * f \in S$ which shows that $k * f \in D(S)$. Linearity is obvious. If $k * f = 0$ for any $f \in S$, then $\hat{k}(u)$ $\hat{f}(u) = 0$ for all $u \in R$. But $\hat{k}(u) = \frac{i}{u - i}$ is non-vanishing for any finite u. Hence $\hat{f}(u) = 0$ for all $u \in R$. Hence $f = 0$. This proves that K is one to one. Now we shall show that K is actually onto. Let $f \in D(S)$. Consider

$$k * \left(\frac{df}{dx} + f\right) = k * \left(\frac{d}{dx} + 1\right) f = \left(\frac{d}{dx} + 1\right) k * f = \delta * f = f.$$

Since $\left(\frac{df}{dx} + f\right) \in S$, the map is onto. The inverse map $K^{-1} : D(S) \to S$ is the differential operator $\left(\frac{d}{dx} + 1\right)$ which is continuous from $D(S)$ to S. By open mapping theorem K is also continuous. This proves the theorem.

Theorem 2.4 — The space of multipliers from S into itself is topologically isomorphic to the space of multipliers from $D(S)$ into $D(S)$.

PROOF : If T is a multiplier on S then it is a multiplier on $D(S)$ since T leaves $D(S)$ invariant. Since $D(S)$ is dense in S, T is determined by its restriction on $D(S)$. Thus the map $T \to T/D(S)$ is one to one. If T_0 is any multiplier from $D(S)$ into itself, consider $T = K^{-1}T_0K : S \to S$ where K is the topological isomorphism defined in the proof of Theorem 2.3. By definition T is continuous from S into itself and commutes with translations. Also, restriction of T to $D(S)$ coincides with T_0. Hence the theorem follows.

As a consequence of the above theorem we have the following multiplier results.

Theorem 2.5 — The space of multipliers on $L^A(R)$ is isomorphic to the space $M(R)$ of bounded measures on R.

PROOF : As already observed $D(L^1) = L^A(R)$. As the multipliers on L^1 are given by bounded measures, the result follows as an easy application of Theorem 2.4.

Proposition 2.6 — On the real line R, there exists an infinite class of Segal algebras whose multiplier spaces strictly contain $M(R)$, the space of bounded measures.

PROOF : Let W be the Wiener algebra. As functions in W are continuous on R, whenever $\frac{df}{dx} \in W$, we have $\frac{df}{dx} = f'$, where f' is the classical derivative of f. Hence we can write $D(W)$ as

$$\{f \in W : f' \in W\}.$$

Put $D_0 = W$ and define for $n \geqslant 1$,

$$D_n = \{f \in D_{n-1} : f' \in D_{n-1}\}.$$

Then D_n is a Segal algebra for each $n \geqslant 1$.

If $\mathcal{E}^n$ (resp. $\mathcal{E}^n_c$) denotes the space of functions (resp. with compact support) of class C^n we have by Schwartz (1966)

$$\mathcal{E}^n_c \subset D_n \subset \mathcal{E}^n,\ n \geqslant 0.$$

By taking a function in $\mathcal{E}^n_c$ which is not in $\mathcal{E}^{n+1}$ we see that $D_{n+1} \subset D_n$ strictly holds. Hence $\{D_n\}_{n \geqslant 1}$ is a sequence of Segal algebras properly contained in W. Also, by Theorem 2.4, the multiplier space of D_n is isomorphic to that of W for each $n \geqslant 1$. Since the Wiener algebra W has unbounded measures also as multipliers (see Krogstad 1976) we get that the multiplier spaces of D_n, $n \geqslant 1$, strictly contain the bounded measures.

Remark 2.7 : Krogstad (1976) has mentioned that the classical Wiener algebra and its generalizations given by him seem to be the only known Segal algebras on non compact groups where the multipliers strictly contain the bounded measures. Here, by means of Proposition 2.6 we have exhibited that there exist infinitely many such Segal algebras distinct from W on the real line itself.

Remark 2.8 : The referee has pointed out that Tewari (1976) has constructed other Segal algebras on noncompact groups whose multipliers strictly contain the measures.

3. Multipliers for W and V

Using the function $k(x)$ it is possible to define isomorphism between spaces other than Segal algebras. For example, we have the following theorem.

Theorem 3.1 — The space $L^1 \cap BV$ is topologically isomorphic to $M(R)$, the space of all bounded measures and $L^1 \cap BV$ can be identified with the space $k * M(R)$.

The proof is analogous to that of Theorem 2.3. Hence we can make the following proposition on multipliers.

Proposition 3.2 — If A and B are subsets of tempered distributions on $\mathbf{R}$, then

(i) $\varphi \in (A, B)$ if and only if $\dfrac{i\varphi(u)}{u-i} \in (A, k * B)$;

(ii) $\varphi \in (k * B, A)$ if and only if $\dfrac{i\varphi(u)}{u-i} \in (B, A)$.

The proof is trivial, noticing that $\hat{k}(u) = \dfrac{i}{u-i}$, $u \in R$.

If $A, B = S$ in the proposition, then we get the result of Theorem 2.4 as $(D(S), D(S)) = (S, S)$.

Now let A be any subset of tempered distributions.

Corollary 3.3 — (i) $\phi \in (A, L^1)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (A, L^A)$;

(ii) $\phi \in (L^A, A)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (L^1, A)$.

Proof : Since $L^A = D(L^1) = k * L^1$ from Theorem 2.5 the corollary follows.

Corollary 3.4 — (i) $\phi \in (A, M)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (A, L^1 \cap BV)$;

(ii) $\phi \in (M, A)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (L^1 \cap BV, A)$.

This follows from Theorem 3.1.

Corollary 3.5 — (i) $\phi \in (A, W)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (A, D_1)$;

(ii) $\phi \in (W, A)$ if and only if $\dfrac{i\phi(u)}{u-i} \in (D_1, A)$.

PROOF : As $D_1 = D(W) = k * W$ from Proposition 2.6, we get the corollary.

This corollary connects all the multipliers of W and D_1. In particular, putting $A = L^1$, we get

$$(L^1, D_1) = k \hat{*} V$$

by observing that $M(L^1, W)$ is given by functions of V (see Burnham and Goldberg 1975).

Proposition 3.2 is actually a generalization of certain multiplier results of Doss (1949). In corollary 3.3, taking A to be the various classes of function F_i, $i=1, 2, \ldots, 6$ considered by Doss (1949), we get the corresponding results given in his paper. He has characterized multipliers ϕ by specifying the class to which $\frac{i\phi(u)}{u - i}$ belongs. Adopting this technique, in the sequel we shall establish equivalences among several multiplier spaces of W and V.

Proposition 3.6 — (i) $\phi \in (L^A, W)$ if and only if $\frac{i\phi(u)}{u - i} \in \hat{V}$;

(ii) $(L^A, W) = (L^A, V) = (L^1 \cap BV, V)$.

PROOF : To get (i), put $A = W$ in Corollary 3.3 (ii) and use the fact that

$$(L^1, W) = \hat{V}. \quad \ldots(7)$$

To get (ii) we notice that $(L^A, W) = (L^A, V)$ is proved in Poornima (1978) and it is enough to show that

$$\phi \in (L^1 \cap BV, V) \text{ if and only if } \frac{i\phi(u)}{u - i} \in \hat{V}.$$

Since $k \in L^1 \cap BV$, for any $\phi \in (L^1 \cap BV, V)$ we get immediately $\phi\hat{k} = \frac{i\phi(u)}{u - i} \in \hat{V}$. To prove the converse, suppose $\frac{i\phi(u)}{u - i} \in \hat{V}$. Then there exists $h \in V$ such that

$$\frac{i\phi(u)}{u - i} = \hat{h}(u), \ u \in R. \quad \ldots(8)$$

Let $f \in L^1 \cap BV$. We claim that $\phi\hat{f} \in \hat{V}$. For any $g \in L^1$ we have $\hat{h}\,\hat{g} \in \hat{W}$ by (7). By Theorem 3.1 there exists $\mu \in M(R)$ such that $f = k * \mu$ so that

$$\hat{f}(u) = \frac{i}{u - i}\,\hat{\mu}(u), \ u \in R. \quad \ldots(9)$$

Hence using (8) and (9) we see that $\phi(u)\,\hat{f}(u)\,\hat{g}(u) = \hat{\mu}(u)\,(\hat{h}(u)\,\hat{g}(u)) \in \hat{W}$ because

$\widehat{M(R)} \subset (W, W)$. Thus for each $g \in L^1$ we have proved that $\phi \hat{f} \hat{g} \in \hat{W}$ which means that $\phi \hat{f} \in (L^1, W) = \hat{V}$ and our claim is established.

Corollary 3.7 — $(L^1, V) = \hat{V}$.

PROOF : Put $A = V$ in Corollary 3.3(ii) and use Proposition 3.6. (ii).

Proposition 3.8 — (i) $(L^1 \cap L^\infty, V) = (L^1 \cap RI, V) = (L^1 \cap C_0, V) = (L^1 \cap C_0, W)$;

(ii) $(L^1 \cap L^\infty, W) = (L^1 \cap RI, W)$.

PROOF : (i) Let $\phi \in (L^1 \cap L^\infty, V)$. Then $\phi \in (L^1 \cap L^\infty, L^1 \cap L^\infty)$. But $(L^1 \cap L^\infty, L^1 \cap L^\infty) = (L^1 \cap C_0, L^1 \cap C_0)$ from Doss (1949). Hence $\phi \in (L^1 \cap L^\infty, V)$ implies that ϕ takes $L^1 \hat{\cap} C_0$ into $\hat{V}$. Thus $\phi \in (L^1 \cap C_0, L^1 \cap C_0 \cap V) = (L^1 \cap C_0, W)$ and we have $(L^1 \cap L^\infty, V) \subset (L^1 \cap C_0, W)$. To prove the opposite inclusion, let $\phi \in (L^1 \cap C_0, W)$ and let $f \in L^1 \cap L^\infty$. Since $L^1 \cap L^\infty = (L^1, L^1 \cap C_0)$ from Doss (1949), we see that $\phi \hat{f} \in (L^1, W) = \hat{V}$ and so $\phi \in (L^1 \cap L^\infty, V)$. By the inclusion relation (1) it is enough to show that $(L^1 \cap C_0, V) = (L^1 \cap C_0, W)$. But

$$(L^1 \cap C_0, V) = (L^1 \cap C_0, W)$$

is proved in Poornima (1978) using the continuously translating property of W. This proves (i).

(ii) It is enough to show that $(L^1 \cap RI, W)$ is contained in $(L^1 \cap L^\infty, W)$. Since $(L^1 \cap RI, W)$ is contained in $(L^1 \cap RI, L^1 \cap C_0)$ which is $(L^1 \cap L^\infty, L^1 \cap C_0)$ by Doss (1949) and $(L^1 \cap RI, W) \subset (L^1 \cap C_0, W) = (L^1 \cap L^\infty, V)$ we get

$$(L^1 \cap RI, W) \subset (L^1 \cap L^\infty, V \cap L^1 \cap C_0) = (L^1 \cap L^\infty, W).$$

Proposition 3.9 — $\phi \in (L^1 \cap BV, W)$ if and only if $\dfrac{i\phi(u)}{u - i} \in \hat{W}$.

PROOF : $\phi \in (L^1 \cap BV, W)$ implies $\dfrac{i\phi(u)}{u-i} \in \hat{W}$, since $k \in L^1 \cap BV$. Now suppose $\dfrac{i\phi(u)}{u-i} \in \hat{W}$. Then $\phi(u) = \dfrac{u-i}{i} \hat{h}(u)$ for some $h \in W$. Let $g \in L^1 \cap BV$. Then there exists $\mu \in M(R)$ such that $g = k * \mu$ by Theorem 3.1. Now

$$\phi(u)\, g(u) = h(u)\, \hat{\mu}(u) \in \hat{W}$$

as $\widehat{M(R)} \subset (W, W)$. Hence $\phi \in (L^1 \cap BV, W)$.

Proposition 3.10 — $(V, L^1 \cap L^\infty) = (W, L^1 \cap L^\infty) = (W, L^1 \cap RI)$

$= (W, L^1 \cap C_0)$.

PROOF : We shall first get $(V, L^1 \cap L^\infty) = (W, L^1 \cap L^\infty)$. Enough to prove $(W, L^1 \cap L^\infty) \subset (V, L^1 \cap L^\infty)$. Let $\phi \in (W, L^1 \cap L^\infty)$, $f \in V$ and $g \in L^1$. Then $\hat{f}\hat{g} \in \hat{W}$ so that $\phi\hat{f}\hat{g} \in L^1 \hat{\cap} L^\infty$. This means that $\phi\hat{f} \in (L^1, L^1 \cap L^\infty)$. But

$$(L^1, L^1 \cap L^\infty) = L^1 \hat{\cap} L^\infty$$

by Doss (1949). Hence $\phi \in (V, L^1 \cap L^\infty)$. That $(W, L^1 \cap L^\infty) = (W, L^1 \cap C_0)$ is proved in Poornima (1978). By the inclusion relation (1) the proposition follows.

Proposition 3.11 — $(V, L^1 \cap BV) = (W, L^A) = (W, L^1 \cap BV)$.

PROOF : In Poornima (1978) using the continuously translating property of L^A and W, it is proved that $(W, L^A) = (W, L^1 \cap BV)$. Hence it is enough to show that $(W, L^A) \subset (V, L^1 \cap BV)$. Let $\phi \in (W, L^A)$ and $f \in V$. Then $\hat{f} \in (L^1, W)$ which in turn gives $\phi\hat{f} \in (L^1, L^A) = L^1 \cap \hat{B}V$ [see Burnham and Goldberg (1975) and Doss (1949)]. Thus $\phi\hat{f} \in (L^1 \cap BV)^\wedge$ for every $f \in V$. Hence $\phi \in (V, L^1 \cap BV)$.

Proposition 3.12 — $(W, W) = (W, V) = (V, V)$.

PROOF : We shall first show that $(V, V) = (W, V)$. Clearly $(V, V) \subset (W, V)$. If $\phi \in (W, V)$, let $f \in V$ and $g \in L^1$. Then $\hat{f}\hat{g} \in \hat{W}$ and so $\phi\hat{f}\hat{g} \in \hat{V}$ which implies $\phi\hat{f} \in (L^1, V) = \hat{V}$ by Corollary 3.7. Hence $(W, V) \subset (V, V)$. That $(W, W) = (W, V)$ is proved in Poornima (1978). Hence the Proposition.

Corollary 3.13 — $(V, L^1 \cap C_0) \cap (W, W) = (V, W)$.

Corollary 3.14 — $(V, k * V) = (W, D_1) = (W, k * V)$.

PROOF : Putting $B = V$, $A = V, W$ in Proposition 3.2(i) and $A = W$ in Corollary 3.5 (i), we get the result.

Proposition 3.15 — (i) $(V, W) \supsetneq L^1(\hat{R})$;

(ii) $(V, W) \cap \widehat{M(R)} = L^1(\hat{R})$.

PROOF : (i) That $L^{\hat{1}}(R) \subset (V, W)$ is clear. Put $X = (V, W)$. Since

$$(V, W) \subset (W, W)$$

in view of Unni (1974), X is a subspace of pseudomeasures on R and hence $\hat{X}$ is meaningful. By the kind of arguments used in the foregoing, it can be proved that

$$(L^1, X) = (V, V).$$

Since $(V, V) = (W, W)$ and $M(R) \subsetneq (W, W)$ (see, for example Krogstad 1976), X cannot be $L^1(R)$.

(ii) Let $\phi \in (V, W) \cap \widehat{M(R)}$. That is, there exists $\mu \in M(R)$ such that $\phi = \hat{\mu}$ and $\mu * f \in W$ for each $f \in V$. If C is any relatively compact set of R, ξ_{-C}, the characteristic function of $-C$ belongs to V. Hence $\mu * \xi_{-C} \in W$. Therefore the function $x \to \mu(x + C)$ is continuous on R. By the result of Pigno (1973) this would imply that μ belongs to $L^1(R)$. Hence we get (ii).

From above we notice that $(V, W) \subsetneq (W, W)$ since $\widehat{M(R)} \subset (W, W)$ Using this we can also obtain the following inclusion relations for the multiplier spaces of the Segal algebra D_1.

Proposition 3.16 — (i) $L^1 \hat{\cap} BV \subsetneq (W, D_1) \subset \hat{V}$;

(ii) $\hat{L}^A \subsetneq (V, D_1) \subset \hat{W}$.

PROOF : (i) Let $\phi \in (W, D_1)$ which is the same as $(V, k * V)$ from Corollary 3.14. Then $\phi\hat{k} = \dfrac{i\phi(u)}{u - i}$ belongs to $k \hat{*} V$, since $k \in V$. Thus $\phi \in \hat{V}$. In Corollary 3.5(i) put $A = W$. Then the fact that $\widehat{M(R)} \subsetneq (W, W)$ and Theorem 3.1, give (i).

(ii) In Corollary 3.5(i) put $A = V$ and use Proposition 3.15(i) and the fact that $L^A = k * L^1$ to get $L^A \subsetneq (V, D_1)$. If $\phi \in (V, D_1)$ then $\dfrac{i\phi(u)}{u - i} \in \hat{D}_1 = k \hat{*} W$. Hence $\phi \in \hat{W}$. This completes the Proposition.

4. Multipliers From $L^1 \cap C_0$ to the Wiener Algebra

In this section we identify all the multipliers spaces (L^1, W), $(L^1 \cap L^\infty, W)$, $(L^1 \cap RI, W)$, $(L^1 \cap C_0, W)$, (L^1, V), $(L^1 \cap L^\infty, V)$, $(L^1 \cap RI, V)$ and $(L^1 \cap C_0, V)$ with the class $\hat{V}$. This turns out to be a consequence of the main result of this section which reveals that the space of multipliers from $L^1 \cap C_0$ into W can be identified with the space V itself. We adopt a technique of Hormander (1960) to arrive at this result. Here, by the term multiplier, we mean a bounded linear transformation which commutes with translations. We recall the following result of Hormander (1960).

Lemma 4.1 — Let $y \in \mathbf{R}^n$ and $f \in L^p(R^n)$. If $1 \leqslant p < \infty$, then

$$\| f + \tau_y f \|_p \to 2^{1/p} \| f \|_p \quad \text{as} \quad y \to \infty.$$

The result is valid for $p = \infty$ whenever f vanishes at infinity.

Its analogue for W, we state as follows.

Theorem 4.2 — Let $f \in W$ and $y \in R$. Then

$$\| f + \tau_y f \|_W \to 2 \| f \|_W \quad \text{as} \quad y \to \infty.$$

PROOF : The set $C_c(R)$ of continuous functions on R with compact support is dense in W. Given $\epsilon > 0$ and $f \in W$ there exists $g \in C_c(R)$ such that $f = g + h$ where $\| h \|_W < \epsilon$ and

$$| \| f \|_W - \| g \|_W | < \epsilon \qquad \ldots(10)$$

$$| \| f + \tau_y f \|_W - \| g + \tau_y g \|_W | < 2\epsilon. \qquad \ldots(11)$$

If y is sufficiently large, then the supports of g and $\tau_y g$ have empty intersection and we have

$$\| g + \tau_y g \|_W = \| g \|_W + \| \tau_y g \|_W = 2 \| g \|_W. \qquad \ldots(12)$$

Since ϵ is arbitrary, (10), (11) and (12) give the desired result.

Theorem 4.3 — The multiplier space $M(L^1 \cap C_0, W)$ coincides with the multiplier space $M(L^1, W)$; and so $M(L^1 \cap C_0, W)$ is isometrically isomorphic to V.

PROOF : Since $L^1 \cap C_0 \subset L^1$, we need to prove only that

$$M(L^1 \cap C_0, W) \subset M(L^1, W).$$

To this end, let $T \in M(L^1 \cap C_0, W)$. Then there exists $C > 0$ such that

$$\| Tf \|_W \leqslant C(\| f \|_1 + \| f \|_\infty), \quad f \in L^1 \cap C_0(R).$$

Since T is linear and translation invariant, for $y \in R$ we have

$$\| Tf + \tau_y Tf \|_W = \| T(f + \tau_y f) \|_W \leqslant C(\| f + \tau_y f \|_1 + \| f + \tau_y f \|_\infty).$$

Letting $| y | \to \infty$ and applying Theorems 4.2 and Lemma 4.1 (for $p = 1$ and ∞) we get

$$2 \| Tf \|_W \leqslant C(2 \| f \|_1 + \| f \|_\infty)$$

so that

$$\| Tf \|_W \leqslant C(\| f \|_1 + 2^{-1} \| f \|_\infty)$$

Repeating the argument n times, this yields

$$\| Tf \|_W \leqslant C \| f \|_1 + 2^{-n} \| f \|_\infty)$$

from which we deduce, letting $n \to \infty$ on the right-hand side,

$$\| Tf \|_W \leqslant C \| f \|_1$$

for all $f \in L^1 \cap C_0$. Since $L^1 \cap C_0$ is dense in $L^1(R)$, T can be extended as a multiplier from L^1 to W. That $M(L^1 \cap C_0, W)$ is isometrically isomorphic to V follows from the result of Burnham and Goldberg (1975).

Corollary 4.4 — $(L^1, W) = (L^1 \cap L^\infty, W) = (L^1 \cap RI, W) = (L^1 \cap C_0, W)$

$$= (L^1, V) = (L^1 \cap L^\infty, V) = (L^1 \cap RI, V) = (L^1 \cap C_0, V) = \hat{V}.$$

PROOF: Since $L^1 \cap C_0 \subset L^1 \cap RI \subset L^1 \cap L^\infty \subset L^1$, by Proposition 3.8(i) and (ii) we get the corollary.

Theorem 4.5 — For $1 \leqslant p < \infty$, the multiplier space $M(L^1 \cap L^p, W)$ is isometrically isomorphic to the space V.

The proof is analogous to that of Theorem 4.3.

Remark 4.6 : By defining the Wiener algebra on $\mathbf{R}^n$ as given in Krogstad (1976), we can extend the results of this section to functions on $\mathbf{R}^n$.

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