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ON SOME THEOREMS OF ISEKI

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The purpose of this paper is to generalize three recent fixed point theorems due to Iseki.

PRELIMINARIES

Let A be a bounded subset of a complete metric space X. By the real number $\alpha(A)$ we denote the infimum of all positive numbers ϵ such that A admits a finite covering consisting of subsets with diameter less than ϵ. If A, B are bounded subsets of X, then

$$0 \leqslant \alpha(A) \leqslant D(A) \text{ where } D(A) \text{ is the diameter of } A, \quad \ldots(1)$$

$$\alpha(A) = 0 \text{ if and only if } A \text{ is precompact}, \quad \ldots(2)$$

$$\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}. \quad \ldots(3)$$

$$\alpha(A) = \alpha(B) \text{ if } A - B \text{ is a finite set where}$$
$$A - B = \{x/x \in A \text{ and } x \notin B\}. \quad \ldots(4)$$

A self mapping T on a metric space X is called densifying if for any bounded subset A of X with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$.

If A is a subset of X where (X, d) is a metric space we denote by $\bar{A}$ the closure of A in (X, d).

MAIN RESULTS

Theorem 1 — Suppose S and T are continuous densifying mappings of a bounded complete metric space X into itself. Suppose F is a real valued continuous function on $X \times X$. If for every x, y in X with $x \neq Sx$ and $y \neq Ty$,

$$\left.\begin{array}{l} F(Sx, Ty) < \frac{1}{2}\,(F(x, Sx) + F(y, Ty)) \\ \text{and} \\ F(Ty, Sx) < \frac{1}{2}\,(F(x, Sx) + F(y, Ty)) \end{array}\right\} \quad \ldots(5)$$

then either S or T has a fixed point.

PROOF : Let A be the subsemigroup generated by the elements S and T in the semigroup of all mappings on X with composition of mappings as the binary relation. Let x be a fixed but arbitrary element of X.

$$\text{Let } M = M(x) = \left\{ \begin{array}{l} y \in X/y = g(x) \text{ for some g} \\ \text{in } A \text{ or } y = x. \end{array} \right\}$$

Then, clearly $S(M) \subseteq M$ and $T(M) \subseteq M$. By the continuity of S and T we have $S(\bar{M}) \subseteq \bar{M}$ and $T(\bar{M}) \subseteq \bar{M}$. Also $M = S(M) \cup T(M) \cup \{x\}$. Hence

$$\alpha(M) = \max \{\alpha(S(M)), \alpha(T(M))\}.$$

Since S and T are densifying and since M is invariant under S and T it follows that $\alpha(M) = 0$. Hence M is precompact. Since X is complete, it follows that $\bar{M}$ is compact. Define the function f by

$$f(x) = \min \{F(x, Sx), F(x, Tx)\}. \quad \ldots(6)$$

Since $\bar{M}$ is compact, f attains its minimum on $\bar{M}$ at some point z in $\bar{M}$. If $Sz = z$ or $TSz = Sz$ or $Tz = z$ or $STz = Tz$, the theorem is true. Suppose, to obtain a contradiction, that none of the above four equations is true. Now we consider two cases.

Case (*i*) — $f(z) = F(z, Sz)$. Then by (5)

$$F(Sz, TSz) < \tfrac{1}{2} (F(z, Sz) + F(Sz, TSz))$$

so that

$$F(Sz, TSz) < F(z, Sz) = f(z).$$

But by definition of f, it follows that $f(Sz) \leqslant F(Sz, TSz) < F(z, Sz) = f(z)$ contradicting the minimality of $f(z)$.

Case (*ii*) — $f(z) = F(z, Tz)$. Then by (5)

$$F(Tz, STz) < \tfrac{1}{2} (F(z, Tz) + F(Tz, STz))$$

so that

$$F(Tz, STz) < F(z, Tz).$$

Hence $\qquad f(Tz) \leqslant F(Tz, STz) < F(z, Tz) = f(z).$

This is a contradiction to the minimality of $f(z)$.

This completes the proof of Theorem 1.

Remark 1* : This theorem remains valid when the boundedness of X is replaced by the boundedness of $M(x)$ for some x in X.

If d is a metric on X then d is continuous on $X \times X$. Hence we have the following corollary to Theorem 1.

*The author is thankful to the referee for suggesting this remark.

Corollary 1 — Suppose S and T are continuous densifying mappings of a bounded complete metric space (X, d) into itself. If for every x, y in X with $x \neq Sx$ and $y \neq Ty$,

$$d(Sx, Ty) < \tfrac{1}{2} (d(x, Sx) + d(y, Ty))$$

then either S or T has a fixed point.

Example — Let $X = \{0, 1\}$. Let the metric d be defined on X by

$$d(x, y) = | x - y |$$

for x, y in X. Let S and T be defined by $S(0) = T(1) = 0$ and $S(1) = T(0) = 1$. Then all the conditions of Corollary 1 are satisfied. Now S has 0 and 1 as its fixed points. T has no fixed points. This example shows that the conclusion of Corollary 1 (and hence Theorem 1) is the best possible.

However, a slight modification of the hypothesis of Corollary 1 will ensure a fixed point for both S and T as we shall see in Theorem 2. In fact we can obtain a common fixed point for S and T.

Theorem 2 — Suppose S and T are continuous densifying mappings of a bounded complete metric space (X, d) into itself. If for every x, y in X with $x \neq Sx$ or $y \neq Ty$,

$$d(Sx, Ty) < \tfrac{1}{2} (d(x, Sx) + d(y, Ty)) \qquad \ldots(7)$$

then $\phi \neq F_{S,T} = F_S = F_T$ where $F_{S,T} = \{x \in X/Sx = Tx = x\}$,

$$F_S = \{x \in X/Sx = x\} \text{ and } F_T = \{x \in X/Tx = x\}.$$

PROOF : Clearly $F_{S,T} \subseteq F_S$ and $F_{S,T} \subseteq F_T$. Suppose $x \in F_S$. Then $Sx = x$. Suppose, if possible, $Tx \neq x$. Then by (7)

$$d(x, Tx) = d(Sx, Tx) < \tfrac{1}{2} (d(x, Sx) + d(x, Tx)) = \tfrac{1}{2} d(x, Tx).$$

This is a contradiction. Hence $F_S \subseteq F_{S,T}$ and $F_S = F_{S,T}$. Similarly $F_T = F_{S,T}$. Nonemptiness of the sets follows from Corollary 1.

Theorem 3 — Let S and T be continuous densifying mappings of a bounded complete metric space (X, d) into itself. Let F_S, F_T and $F_{S,T}$ be as in Theorem 2. If for every distinct x, y in X

$$d(Sx, Ty) < \tfrac{1}{2} (d(x, Ty) + d(y, Sx)) \qquad \ldots(8)$$

then either S or T has a fixed point, Also if both F_S and F_T are nonempty then $F_{S,T} = F_S = F_T = \{x\}$ for some x.

PROOF : Let the set M be as in Theorem 1. The function

$$f(x) = \min \{d(Sx, TSx), d(Tx, STx)\}$$

is continuous on $\bar{M}$. Since $\bar{M}$ is compact, there is a z in $\bar{M}$ such that $f(z) \leqslant f(x)$ for all x in $\bar{M}$. If $Sz = TSz$ or $Tz = STz$ we are through. Suppose to obtain a contradiction $Sz \neq TSz$ and $Tz \neq STz$.

Case (*i*) : $f(z) = d(Sz, TSz)$ — Then by (8) and the triangle inequality,

$$d(TSz, STSz) < \tfrac{1}{2}\, d(Sz, STSz) \leqslant \tfrac{1}{2}(d(Sz, TSz) + d(TSz, STSz)).$$

Hence $f(Sz) \leqslant d(TSz, STSz) < d(Sz, TSz) = f(z)$, contradicting the minimality of z.

Case (*ii*) : $f(z) = d(Tz, STz)$ — Then by (8) and the triangle inequality,

$$d(STz, TSTz) < \tfrac{1}{2}\, d(Tz, TSTz) \leqslant \tfrac{1}{2}\,(d(Tz, STz) + d(STz, TSTz)).$$

Hence $f(Sz) \leqslant d(STz, TSTz) < d(Tz, STz) = f(z)$, contradicting the minimality of z. Hence either F_S or F_T is nonempty.

Suppose $F_S, F_T \neq \phi$. Suppose $x \in F_S$ and $y \in F_T$ and $x \neq y$. Then

$$d(x, y) = d(Sx, Ty) < \tfrac{1}{2}\,(d(x, Ty) + d(y, Sx))$$
$$= \tfrac{1}{2}\,(d(x, y) + d(x, y)) = d(x, y),$$

which is a contradiction. Hence $F_S = F_T = \{x\}$ for some x in X. This completes the proof of Theorem 3.

Example — The example following Corollary 1 also satisfies the hypothesis of Theorem 3. Here $F_S = \{0, 1\}$ where $F_T = \phi$.

As a corollary to Theorem 3 we state the following

Corollary 2 — If in Theorem 3, $S = T$ then S has a unique fixed point.

Theorem 4 — Let S and T be continuous densifying mappings of a bounded complete metric space (X, d) into itself. If for all x, y in X with $Sx \neq Ty$,

$$d(Sx, Ty) < \tfrac{1}{2}\,(d(x, Ty) + d(y, Sx)) \text{ then } F_{S,T} = F_S = F_T = \{x\} \quad \ldots(9)$$

for some x in X.

PROOF : First we claim that either $F_S \neq \phi$ or $F_T \neq \phi$. Let M, f and z be as in Theorem 3. If $TSz = STSz$ or $STz = TSTz$, then the claim is true. Suppose, to obtain a contradiction, $TSz \neq STSz$ and $STz \neq TSTz$.

Case (*i*) : $f(z) = d(Sz, TSz)$ — Then by (9) and the triangle inequality,

$$d(TSz, STSz) < \tfrac{1}{2}\, d(Sz, STSz) \leqslant \tfrac{1}{2}\,(d(Sz, TSz) + d(TSz, STSz).$$

Hence $f(Sz) \leqslant d(TSz, STSz) < d(Sz, TSz) = f(z)$, contradicting the minimality of z.

Case (*ii*) : $f(z) = d(Tz, STz)$ — A contradiction is obtained in this case also by a similar reasoning.

Hence either S or T has a fixed point.

Suppose $Sx = x$ and $Tx \neq x$. Then by (9)

$$d(x, Tx) = d(Sx, Tx) < \tfrac{1}{2}(d(x, Tx) + d(x, Sx)) = \tfrac{1}{2} d(x, Tx)$$

which is a contradiction. Thus $Tx = x$ and $F_S \subseteq F_T$. Similarly $F_T \subseteq F_S$. Hence $F_S = F_T$.

Now suppose $Sx = Tx = x$ and $Sy = Ty = y$ and $x \neq y$. Then

$$d(x, y) = d(Sx, Ty) < \tfrac{1}{2}(d(x, Ty) + d(y, Sx)) = d(x, y),$$

a contradiction. Therefore $F_{S,T} = F_S = F_T = \{x\}$. This completes the proof of Theorem 4.

Corollary 3 — If in Theorem 4, $S = T$, then S has a unique fixed point.

Remark 2 : Corollary 2 and Corollary 3 are equivalent.

Remark 3 : Theorem 1 and Corollary 1 with the assumption $S = T$ and also Corollary 2 have been proved by Iseki.

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