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ON SOME ANALYTIC PROPERTIES OF THE MODIFIED SINE OPERATOR

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In this paper we have defined the modified sine operator family associated with a regular semigroup of operators on a Banach space and studied the regularity properties in the strong operator topology. It is established that the measurability of the modified sine operator family implies the continuity of the family. Moreover, the continuity of the modified sine operator at the origin implies continuity everywhere. A bound for the norm of the operators in the family has also been obtained. Some results regarding the commutativity of the family are also established.

1. INTRODUCTION

The sine functional equation

$$\left.\begin{aligned} f(t+s)\,f(t-s) &= f^2(t) - f^2(s) \\ f(0) &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

where s, t are real numbers, and f is a real valued function of a real variable, is well known (cf. Aczél 1966). The most general continuous solutions of (1) are $\sin a\,t$, $\sinh a\,t$, $a\,t$ where a is a scalar. An analogue of (1) in the n-dimensional vector space, f denoting a square matrix of order n, was considered by Kurepa (1960), who has discussed the connection between the Lebesgue measurability and differentiability of the solutions.

Let X be a Banach space, and $B(X)$ the family of bounded linear operators on X. Let $R^+ = [0, \infty)$, and I denote the identity operator.

Buche and Bharucha-Reid (1968) considered a pair of operator-valued functional equations

$$\left.\begin{aligned} S(s+t) - S(s)\,S(t) &= U(s)\,U(t) \\ U(s+t) - S(s)\,U(t) &= U(s)\,S(t) \end{aligned}\right\} \qquad \ldots(2)$$

where $\{S(t);\, t \in R^+\}$, $S : R^+ \to B(X)$, $S(0) = I$, and $\{U(t);\, t \in R^+\}$, $U : R^+ \to B(X)$, $U(0) = 0$, are one parameter families of operators. The solutions of (2) in the strong operator topology were discussed by Buche (1971), and are given by

$$S(t) = (V(t) + W(t))/2, \quad U(t) = (V(t) - W(t))/2$$

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where $\{V(t)\}$, $\{W(t)\}$ are semigroups of operators of class (C_0) (ref. Hille and Phillips 1957). The solution of (2) for $U(t)$ may be called an 'exponential-sine operator', in view of the representation given by Theorem 1 of Buche (1971).

In this paper we consider some properties of a family of operators, which can be considered as a modified version of (1) as well as (2). The 'modified sine operator' is a one parameter family of operators $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$ such that

$$U(t+s) + U(t-s) + 2U(t)\,U(s) = 2U(t)\,T(s) \qquad ...(3)$$

$s, t \in R^+$, $s \leqslant t$, where $\{T(t); t \in R^+\}$, $T : R^+ \to B(X)$, is a known semigroup of operators of class (C_0). We shall study the solution of eqn. (3) in the strong operator topology.

In section 2 we shall discuss the measurability, boundedness and continuity properties of the modified sine operator. Some results about the commutativity of $\{U(t)\}$ and $\{T(t)\}$ are obtained in section 3. Section 4 contains some examples of the modified sine operator.

In subsequent papers, we shall discuss the differential and integral equations associated with the modified sine operator and also the associated resolvent family and the generation theorem for the modified sine operator.

2. The Measurability, Boundedness and Continuity Properties

Proposition 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, satisfy the eqn. (3). Then the Lebesgue measurability of $U(\cdot)\,f$ on $R^+ - \{0\}$ implies continuity of $U(\cdot)\,f$ on $R^+ - \{0\}$.

Proof : To prove this result we shall be first proving that the Lebesgue measurability of $U(\cdot)\,f$ on $R^+ - \{0\}$ implies the boundedness of $U(\cdot)\,f$ on every compact subset of $R^+ - \{0\}$, for a fixed $f \in X$. Then we shall use the boundedness of $U(\cdot)\,f$ to prove its continuity on $R^+ - \{0\}$.

The Lebesgue measurability of $U(\cdot)\,f$ on $R^+ - \{0\}$ implies the Lebesgue measurability of $\| U(\cdot)\,f \|$ on $R^+ - \{0\}$ (cf. Hille and Phillips 1957). Since $\{T(t); t \in R^+\}$ is a (C_0)-semigroup of operators, there exist nonnegative constants M_1 and ω such that

$$\| T(t) \| \leqslant M_1 \exp(\omega t), \; t \in R^+ \qquad ...(4)$$

(Hille and Phillips 1957).

To prove that $\| U(\cdot)\,f \|$ is bounded on every compact subset of $R^+ - \{0\}$, it would suffice to prove that $\| U(\cdot)\,f \|$ is bounded on every closed interval $[a, b] \subset R^+ - \{0\}$. Suppose this is not true, then there exists a $\tau \in [a, b]$ and a sequence $\tau_n \in [a, b]$, such that $\tau_n \to \tau$ and

$$\| U(\tau_n)\,f \| \geqslant n, \qquad n = 1, 2, 3, \ldots . \qquad ...(5)$$

On the other hand $\| U(\cdot) f \|$ being measurable, there exists a constant $M_2 > 0$ and a Lebesgue measurable set $G \subset [0, \tau]$ with

$$m(G) > \frac{3\tau}{4} \qquad \ldots(6)$$

(m denotes Lebesgue measure) such that

$$\sup_{t \in G} \| U(t) f \| \leqslant M_2. \qquad \ldots(7)$$

Let

$$A_k = \frac{\tau_k}{2} - \frac{G \cap [0, \tau_k]}{2},$$

and

$$B_k = G \cap \left[0, \frac{\tau_k}{2}\right].$$

Define

$$A = \frac{\tau}{2} - \frac{G \cap [0, \tau]}{2}$$

and

$$B = G \cap [0, \tfrac{1}{2}\tau].$$

We claim that $m(A \cap B) > 0$. If possible, suppose that $m(A \cap B) = 0$, then $m(A) + m(B) \leqslant \frac{1}{2}\tau$, because $A \subset [0, \frac{1}{2}\tau]$ and $B \subset [0, \frac{1}{2}\tau]$. But $m(A) = \frac{m(G)}{2}$. Since $m(A) + m(B) \leqslant \frac{1}{2}\tau$, we have $m(G) + 2m(B) \leqslant \tau$. Hence $\frac{3}{4}\tau < m(G) \leqslant \tau - 2m(B)$, this is,

$$m(B) < \frac{\tau}{8}. \qquad \ldots(8)$$

Now $\quad G = (G \cap [0, \frac{1}{2}\tau]) \cup (G \cap [\frac{1}{2}\tau, \tau])$

$\qquad = B \cup A_1 \quad$ (say)

implies that

$$m(G) = m(B) + m(A_1)$$

where $m(A_1) \leqslant \frac{1}{2}\tau$.

Hence

$$\tfrac{3}{4}\tau < m(G) = m(B) + m(A_1) \leqslant m(B) + \tfrac{1}{2}\tau$$

that is,

$$m(B) > \tfrac{1}{4}\tau. \qquad \ldots(9)$$

But (8) and (9) contradict each other, hence our claim is proved, that is, $m(A \cap B) > 0$. Then there exists a real $\delta > 0$ such that $m(A \cap B) > \delta$.

Let

$$E = A \cap B, \quad E_n = A_n \cap B_n$$

and
$$H_n = \{\tau_n - \eta;\ \eta \in E_n\}, \quad n = 1, 2, 3, \ldots . \qquad \ldots(10)$$

Now $E_n \to E$; therefore, for sufficiently large n, $m(E_n) > 0$ and for such values of n, if $\eta \in E_n$, one can see easily that η and $\tau_n - 2\eta$ both belong to G. Similar result holds for E and G in place of E_n and G.

Clearly H_n is measurable, and for sufficiently large n, $m(H_n) \geqslant \frac{1}{2}\delta$. Now for $\eta \in E_n$, we have, using eqn. (3),

$$n \leqslant \| U(\tau_n) f \| \leqslant M_3 \| U(\tau_n - \eta) \| + M_2$$

where $M_3 = 2M_2 + 2M_1 \exp(\omega b)$.

Hence
$$\frac{n - M_2}{M_3} \leqslant \| U(\tau_n - \eta) \|, \text{ for } \eta \in E_n$$

that is,
$$\frac{n - M_2}{M_3} \leqslant \| U(t) \|, \text{ for } t \in H_n.$$

Denoting lim sup H_n by H, we see that $m(H) \geqslant \frac{1}{2}\delta$ and $\| U(t) \| = \infty$ for all $t \in H$, which contradicts that $\{U(t);\ t \in R^+\}$ is a family of bounded linear operators.

This establishes the boundedness of $\| U(t) f \|$ on every closed interval of $R^+ - \{0\}$, for every $f \in X$. By the uniform boundedness principle, $\| U(t) \|$ is bounded on every closed interval of $R^+ - \{0\}$.

We shall now prove that the Lebesgue measurability together with boundedness, implies the continuity of the operator family $\{U(t) f\}$ for each $t > 0$ and for each $f \in X$. We choose four numbers α, β, ϵ and η such that $0 < \alpha < \eta < \beta < t$ and ϵ so small that $\beta < t - \epsilon$. We have

$$U(t) f = 2U(t - \tfrac{1}{2}\eta)\, T(\tfrac{1}{2}\eta) f - 2U(t - \tfrac{1}{2}\eta)\, U(\tfrac{1}{2}\eta) f - U(t - \eta) f$$

by eqn. (3). The right-hand side being independent of η, is certainly integrable with respect to η, so that

$$\begin{aligned}
&(\beta - \alpha)\,(U(t \pm \epsilon) f - U(t) f) \\
&= \int_\alpha^\beta [\{2(U(t \pm \epsilon - \tfrac{1}{2}\eta) - U(t - \tfrac{1}{2}\eta))\, T(\tfrac{1}{2}\eta) f\} \\
&\qquad - \{2(U(t \pm \epsilon - \tfrac{1}{2}\eta) - U(t - \tfrac{1}{2}\eta))\, U(\tfrac{1}{2}\eta) f\} \\
&\qquad - \{U(t \pm \epsilon - \eta) f - U(t - \eta) f\}]\, d\eta.
\end{aligned}$$

If $\| U(\frac{1}{2}\eta) f \| \leqslant M_1$ and $\| T(\frac{1}{2}\eta) f \| \leqslant M_2$, for $\alpha \leqslant \eta \leqslant \beta$, the norm of the integrand does not exceed

$$2(M_2 + M_1) \| U(t \pm \epsilon - \tfrac{1}{2}\eta) - U(t - \tfrac{1}{2}\eta) \| + \| U(t \pm \epsilon - \eta) - U(t - \eta) \| \| f \|$$

which is a bounded measurable function of η in $[\alpha, \beta]$. Hence

$$(\beta - \alpha) \| U(t \pm \epsilon) f - U(t) f \| \leqslant 2(M_2 + M_1) \int_{t-\frac{1}{2}\beta}^{t-\frac{1}{2}\alpha} \| U(\tau \pm \epsilon) - U(\tau) \| \, d\tau + \int_{t-\beta}^{t-\alpha} \| U(\tau \pm \epsilon) - U(\tau) \| \| f \| \, d\tau.$$

By the Theorem 3.8.3 of Hille and Phillips (1957), the right-hand side tends to zero with ϵ. It follows that $\{U(t) f\}$ is continuous for $t > 0$, and the proposition is proved.

Definition — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a modified sine operator. Then $\{U(t); t \in R^+\}$ is regular if $\lim U(t) f = 0$ as $t \to 0$, $t \in R^+$, for every $f \in X$.

Proposition 2 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator on X. Then there exist two nonnegative constants M and ω such that $\| U(t) \| \leqslant M \exp(\omega t)$, for every $t \in R^+$.

PROOF : Since $\lim U(t) f = 0$ as $t \to 0$, $t > 0$, and $\lim T(t) f = f$ as $t \to 0$, $t > 0$, by the uniform boundedness principle, there exist an $\eta > 0$, $0 < \eta \leqslant 1$ and $K \geqslant 0$ such that for $0 \leqslant s \leqslant \eta$, $\| U(s) \| \leqslant K$, and $\| T(s) \| \leqslant K$. Let

$$m_1 = \sup_{0 \leqslant s \leqslant \eta} \| U(s) \|, \text{ and } m_2 = \sup_{0 \leqslant s \leqslant \eta} \| T(s) \|.$$

Let $m = \max(m_1, m_2)$. Evidently m is finite and $m \geqslant 1$. One can easily see by induction, using eqn. (3), that $\| U(ns) \| \leqslant (5m)^n$, for every $n = 1, 2, 3, \ldots$, for all $0 \leqslant s \leqslant \eta$.

Now for each $t \in R^+$, there exists a positive integer n, such that $(n - 1)\eta < t \leqslant n\eta$. Since $0 \leqslant t/n \leqslant \eta$ and $n - (t/\eta) \leqslant 1$, it follows that

$$\| U(t) \| \leqslant (5m)^n = (5m)^{t/\eta} (5m)^{n-(t/\eta)} \leqslant (5m) (5m)^{t/\eta}.$$

Define $M = 5m$, and $\omega = 1/\eta \log(5m)$. Then for each $t \in R^+$, $\| U(t) \| \leqslant M \exp(\omega t)$, and this establishes the proposition.

Corollary 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then there exist two nonnegative constants M_1 and ω such that

$$\| U(t) \| \leqslant M_1 \cosh(\omega t), \; t \in R^+.$$

Proposition 3 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then $U(\cdot) f$ is continuous on R^+, for every $f \in X$.

PROOF : We proceed indirectly and assume that there exist $f_0 \in X$ and $t_0 \in R^+$ such that $U(t) f_0$ is not continuous at t_0. Let us define for $n = 1, 2, 3, \ldots$,

$$K_n = \sup \{\| U(t) f_0 - U(s) f_0 \|, \; | t - t_0 | \leqslant t_0/8n, \; | s - t_0 | \leqslant t_0/8n, t, s \in R^+\}.$$

$\{K_n\}$ is a nonincreasing sequence of nonnegative real numbers. Hence there exists $K \in R^+$ such that $K_n \to K$ as $n \to \infty$. Our assumption of discontinuity of $U(\cdot) f_0$ at t_0 implies that $K > 0$. By the definition of K_n, it follows that there exist two sequences $\tau_n \in R^+$, $\sigma_n \in R^+$, $\tau_n < \sigma_n$, $| \tau_n - t_0 | \leqslant t_0/8n$, $| \sigma_n - t_0 | \leqslant t_0/8n$ and $\| U(\tau_n) f_0 - U(\sigma_n) f_0 \| \geqslant K_n - n^{-1}$, for every $n = 1, 2, 3, \ldots$. From the above it is clear that

$$\sigma_n - \tau_n \leqslant t_0/4n \text{ and } | 2\tau_{4n} - \sigma_{4n} - t_0 | \leqslant t_0/8n.$$

So by the definition of K_n,

$$\| U(\sigma_{4n}) f_0 - U(2\tau_{4n} - \sigma_{4n}) f_0 \| \leqslant K_n, \text{ for all } n = 1, 2, 3, \ldots .$$

By (3), we have

$$2(U(t + s) - U(t)) - (U(t + s) - U(t - s)) = 2U(t)\,(T(s) - I) - 2U(t)\,U(s)$$

for every $t, s \in R^+$, $s < t$. Hence

$$2 \| U(\sigma_{4n}) f_0 - U(\tau_{4n}) f_0 \| \leqslant 2 \| U(\tau_{4n}) \| \, \| T(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \| + \| U(\sigma_{4n}) f_0 - U(2\tau_{4n} - \sigma_{4n}) f_0 \| + 2 \| U(\tau_{4n}) \| \, \| U(\sigma_{4n} - \tau_{4n}) f_0 \|.$$

Therefore

$$2\left(K_{4n} - \frac{1}{4n}\right) \leqslant 2 \| U(\tau_{4n}) \| \, \| T(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \| + K_n + 2 \| U(\tau_{4n}) \| \, \| U(\sigma_{4n} - \tau_{4n}) f_0 \|$$

that is,

$$K_{4n} + (K_{4n} - K_n) \leqslant 1/2n + 2 \| U(\tau_{4n}) \| \, \| T(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \| + 2 \| U(\tau_{4n}) \| \, \| U(\sigma_{4n} - \tau_{4n}) f_0 \|.$$

Now by the Proposition 2, $\| U(\tau_{4n}) \|$ is bounded, but $\| T(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \|$ and $\| U(\sigma_{4n} - \tau_{4n}) f_0 \|$ tend to zero as $n \to \infty$, because of the regularity of $T(\cdot)$ and $U(\cdot)$. Further $K_{4n} - K_n \to 0$ as $n \to \infty$. Hence letting $n \to \infty$ in the last inequality, we have $K_{4n} \to 0$ as $n \to \infty$, which is a contradiction. Thus we obtain the desired result.

3. Commutativity of $\{U(t)\}$ and $\{T(t)\}$

Lemma 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator, with $\{T(t); t \in R^+\}$ as the associated semigroup of operators. Let $U(t) T(t) = T(t) U(t)$, $t \in R^+$. Then for each $t \in R^+$ and $n = 1, 2, 3, \ldots$, there exist $(n + 1)^2$ constants $\alpha_{ij}^{(n)}$, $i, j = 0, 1, 2, \ldots, n$, such that

$$U(nt) = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{ij}^{(n)} U^i(t) T^j(t).$$

Proof : For $n = 1$, the assertion is true with $\alpha_{00} = 0$, $\alpha_{10} = 1$, $\alpha_{01} = 0 = \alpha_{11}$. For $n = 2$, using (3), we have $U(2t) = 2U(t) T(t) - 2U^2(t)$. Hence the assertion is true for $n = 2$ with $\alpha_{00} = \alpha_{01} = \alpha_{10} = 0$, $\alpha_{11} = 2$, $\alpha_{02} = 0$, $\alpha_{20} = -2$, $\alpha_{12} = \alpha_{21} = \alpha_{22} = 0$. One can prove the above assertion by induction using (3) and $U(t) T(t) = T(t) U(t)$, $t \in R^+$.

Lemma 2 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator, with $\{T(t)\}$ as the associated semigroup. If $U(t_1) T(t_2) = T(t_2) U(t_1)$ for all $t_1, t_2 \in R^+$, then $U(t_1) U(t_2) = U(t_2) U(t_1)$ for all $t_1, t_2 \in R^+$.

Proof : Since $U(t_1) T(t_2) = T(t_2) U(t_1)$ for all $t_1, t_2 \in R^+$, therefore, by the Lemma 1, $U(mt) U(nt) = U(nt) U(mt)$, $m, n = 1, 2, 3, \ldots, t \in R^+$. Now take $t_1 = r/2^k$, $t_2 = s/2^l$, r, k, l, s positive natural numbers. Then

$$U(t_1) U(t_2) = U(r . l/2^k) U(s . 2^{k-l} . l/2^k)$$

where we assume that $k \geqslant l$. Thus, from the above it follows that

$$U(t_1) U(t_2) = U(s . 2^{k-l} . l/2^k) U(r . l/2^k) = U(t_2) U(t_1).$$

Similarly if we assume that $l > k$, we can prove that

$$U(t_1) U(t_2) = U(t_2) U(t_1).$$

Hence $\quad U(t_1) U(t_2) = U(t_2) U(t_1)$

whenever $t_1 = r/2^k$, $t_2 = s/2^l$. But the rationals of this type are dense in R^+ and $U(t)$ in continuous on R^+, it follows that $U(t_1) U(t_2) = U(t_2) U(t_1)$, for all $t_1, t_2 \in R^+$.

4. Examples

Example 1 — Let $C(R)$ denote the class of bounded uniformly continuous real valued functions on the real line, with sup norm. Let $\{T(t); t \in R^+\}$ be the translation semigroup, which is defined as

$$[T(t) f] (x) = f(x + at) \qquad \ldots(11)$$

where $a \neq 0$ is some real constant, $f \in C(R)$ and $x \in R$, the real line.

Let us define the family of bounded linear operators $\{U(t); t \in R^+\}$ on $C(R)$ as follows:

$$[U(t) f](x) = \tfrac{1}{2}[f(x + at) - f(x - at]. \qquad ...(12)$$

It can be easily seen that $\{U(t); t \in R^+\}$ satisfies eqn. (3) with (11) as the associated semigroup. Also $\{U(t)\}$ is regular and satisfies

$$[U(t)\, T(s) f]\,(x) = [T(s)\, U(t) f]\,(x)$$

and

$$[U(t)\, U(s) f]\,(x) = [U(s)\, U(t) f]\,(x)$$

$s, t \in R^+$. Furthermore $\| U(t) \| \leqslant 1$.

Example 2 — Let $\{T(t); t \in R^+\}$, $T : R^+ \to C(R)$, be a (C_0)-semigroup defined as follows:

$$[T(t) f]\,(x) = \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} f(x - k\mu), \;\; \alpha > 0 \qquad ...(13)$$

(cf. Yoshida 1974), where $\mu > 0$ is some constant.

Let us define the family $\{U(t); t \in R^+\}$ as follows: for each $f \in C(R)$ and $x \in R$,

$$[U(t) f]\,(x) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!} f(x - (2k+1)\,\mu) \text{ (cf. Buche 1971).} \qquad ...(14)$$

One can easily check that $\{U(t); t \in R^+\}$ in a regular modified sine operator with (13) as the associated semigroup. Furthermore, $[U(t)\, T(s) f]\,(x) = [T(s)\, U(t) f]\,(x)$, and $[U(t)\, U(s) f]\,(x) = [U(s)\, U(t) f]\,(x)$, $t, s \in R^+$, $f \in C(R)$, $x \in R$, and $\| U(t) \| \leqslant \exp(\alpha t)$.

Example 3 — Let $\{T(t); t \in R^+\}$ be a semigroup of linear bounded operators on $L_p(-\pi, \pi)$, $1 \leqslant p \leqslant \infty$, or on $C(-\pi, \pi)$,

$$[T(t) f]\,(x) = \sum_{-\infty}^{\infty} e^{\lambda_n t} f_n e^{inx} \qquad ...(15)$$

where $f(x) = \sum_{-\infty}^{\infty} f_n e^{inx}$ is the complex Fourier series, and $i = \sqrt{-1}$. Let us define

$$[U(t) f]\,(x) = \sum_{-\infty}^{\infty} \sinh(\lambda_n t)\, f_n e^{inx} \qquad ...(16)$$

where $f(x) = \sum_{-\infty}^{\infty} f_n e^{inx}$ (cf. Hille and Phillips 1957). Then one can easily observe all the properties as discussed in Examples 1 and 2.

Example 4 — Let X be a Banach space, and $\{T(t); t \in R\}$ be a regular group of operators on X. Then it is known that $T(-t) = [T(t)]^{-1}$, (cf. Hillle and Phillips 1957). Define $U(t) = [T(t) - T(-t)]/2, t \in R^+$. Then it is easy to see that $\{U(t); t \in R^+\}$ is a regular modified sine operator.

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