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ON THE DIFFERENTIAL AND INTEGRAL EQUATIONS ASSOCIATED WITH THE MODIFIED SINE OPERATOR

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In this paper we have defined the first and second infinitesimal generators of the modified sine operator family on a Banach space and studied the relations between these generators and the infinitesimal generator of the associated regular semigroup. We have obtained the differential equation, involving the second infinitesimal generator, associated with the modified sine operator family. Furthermore, we have established an integral equation, in terms of the first and second infinitesimal generators, associated with the modified sine operator family. Finally we have obtained the representation of modified sine operator family in the uniform operator topology.

1. INTRODUCTION

Let X be a Banach space and let $B(X)$ be the space of bounded linear operators on X. Let $R^+ = [0, \infty)$. A one parameter family $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, is called a modified sine operator if

$$U(t + s) + U(t - s) + 2U(t)\, U(s) = 2U(t)\, T(s) \qquad \text{...(1)}$$

$s \leqslant t$, $s, t \in R^+$, $U(0) = 0$, where $\{T(t);\ t \in R^+\}$, $T : R^+ \to B(X)$, is a known (C_0)-semigroup of operators (Ramesh Chander and Buche 1981). It has been proved (Ramesh Chander and Buche 1981) that the continuity of $\{U(t)\}$ at the origin implies:

(i) its continuity everywhere, and

(ii) there exist two nonnegative constants M and ω such that

$$\| U(t) \| \leqslant M \exp(\omega t),\ t \in R^+. \qquad \text{...(2)}$$

In this paper we shall assume that $U(t)\, T(s) = T(s)\, U(t)$, and $U(t)\, U(s) = U(s)\, U(t)$, for all $s, t \in R^+$.

In section 2 we have discussed the properties of the two infinitesimal generators of the modified sine operator.

The 'first infinitesimal generator' E of $\{U(t)\}$ satisfying (1) is defined by

$$Ef = \lim_{h \to 0} E_h f = \lim_{h \to 0} \frac{U(h)\, f}{h},\ f \in D(E),\ h > 0 \qquad \text{...(3)}$$

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where $D(E) \subset X$ and $D(E)$ is the set of elements $f \in X$, for which the limit exists. Clearly $D(E)$ is a linear subspace of X, and E is a linear operator.

The 'second infinitesimal generator' F of $\{U(t)\}$ satisfying (1) is defined by

$$Ff = \lim_{h \to 0} F_h f = \lim_{h \to 0} \frac{2(T(h)f - U(h)f - f)}{h^2}, \quad f \in D(F),\ h > 0 \quad \ldots(4)$$

where $D(F) \subset X$, and $D(F)$ is the set of elements $f \in X$ for which the limit exists. Clearly $D(F)$ is a linear subspace of X, and F is a linear operator. It turns out that

$$Ff = E^2 f = A^2 f,\ f \in D(F),$$

where A is the infinitesimal generator of the associated semigroup, defined as

$$Af = \lim_{h \to 0} A_h f = \lim_{h \to 0} \frac{T(h)f - f}{h}, \quad f \in D(A),\ h > 0 \quad \ldots(5)$$

where $D(A) \subset X$, and $D(A)$ is the set of element $f \in X$ for which the limit exists (ref. Hille and Phillips 1957). Further it is found that $Ef = Af$, $f \in D(F)$ and F happens to be closable. Under nice conditions, it turns out that $\bigcap_{r=1}^{\infty} D(F^r)$ is dense in X. We also obtain the differential and integral equations involving E and F, which are analogous to those obtained for the cosine operator by Sova (1966).

In section 3 we give a representation theorem for the sine operator in the uniform operator topology.

The section 4 contains some examples.

In a subsequent paper we shall be discussing the resolvent theory and the generation theorem for the modified sine operator

2. The Infinitesimal Generators and the Associated Differential and Integral Equations

Proposition 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then

(i) for $f \in D(E)$, $t \in R^+$, $U(t) f \in D(E)$ and

$$EU(t) f = U(t) Ef;$$

(ii) for $f \in D(F)$, $t \in R^+$, $U(t) f \in D(F)$, and

$$FU(t) f = U(t) Ff;$$

(iii) if F exists, so do A^2 and E^2, and

$$Ff = A^2 f = E^2 f,\ f \in D(F);$$

(iv) E and A agree on the domain of F, that is,

$$Ef = Af, f \in D(F);$$

(v) if F exists, F is closable.

PROOF : (i) and (ii) immediately follows from (2) and the commutativity of $U(t)$ and $T(s)$, $U(t)$ and $U(s)$.

(iii) (a) Using (1) for $t = s = h$, $h > 0$ we have for $f \in D(F)$,

$$F_h f = \tfrac{1}{2} A_{h/2}^2 f + \tfrac{1}{2} F_{h/2} f - \tfrac{1}{2} U(\tfrac{1}{2} h) F_{h/2} f.$$

Taking limits on both sides as $h \to 0$, we get

$$Ff = \tfrac{1}{2} \lim_{h \to 0} A_{h/2}^2 f + \tfrac{1}{2} Ff;$$

that is, $f \in D(A^2)$, and $Ff = A^2 f$, for $f \in D(F)$.

(b) Using (1) for $t = s = h$, $h > 0$, we have, for $f \in D(F)$,

$$F_h f = \tfrac{1}{4}(T(\tfrac{1}{2} h) - U(\tfrac{1}{2} h) - I) F_{h/2} f + \tfrac{1}{2} F_{h/2} f + \tfrac{1}{2} E_{h/2}^2 f.$$

Taking limits on both sides, we get

$$Ff = \tfrac{1}{2} Ff + \tfrac{1}{2} \lim_{h \to 0} E_{h/2}^2 f$$

that is, $f \in D(E^2)$, and $Ff = E^2 f$, for $f \in D(F)$.

(iv) For $f \in D(F)$, $\left\| \frac{2(T(h) f - U(h) f - f)}{h^2} \right\|$ is bounded and by (iii), $f \in D(E^2)$, and $f \in D(A^2)$, that is, $f \in D(E)$ and $f \in D(A)$. Now, for $f \in D(F)$,

$$\left\| \frac{T(h) f - U(h) f - f}{h} \right\| = \frac{h}{2} \left\| \frac{2(T(h) f - U(h) f - f)}{h^2} \right\|$$

therefore,

$$\lim_{h \to 0} \left(\frac{T(h) f - f}{h} \right) - \left(\frac{U(h) f}{h} \right) = 0$$

that is, $\quad Af = Ef, \quad$ for $f \in D(F)$.

(v) It can be easily seen using the fact that A^2 is a closed operator (cf. Hille and Phillips 1957).

Lemma 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then

(i) for $0 < h < t$,

$$\frac{1}{h^2}\int_t^{t+h} (t-p)\,U(p)\,f\,dp - \int_{t-h}^{t} (t-p)\,U(p)\,f\,dp \to -\,U(t)\,f$$

as $h \to 0$, for every $f \in X$ and $t \in R^+$;

(ii) for any $f \in X$,

(a) $$\frac{2}{h^2}\int_0^h pU(p)\,f\,dp \to 0, \quad \text{as} \quad h \to 0,\ h > 0,$$

(b) $$\frac{1}{h}\int_0^h U(p)\,f\,dp \to 0, \quad \text{as} \quad h \to 0,\ h > 0,$$

and (c) $$\frac{1}{h}\int_{t-h}^{t+h} U(p)\,f\,dp \to 2U(t)\,f, \quad \text{as} \quad h \to 0,\ h > 0,$$

(d) $$\frac{2}{h^2}\,T(h)\int_0^h pU(p)\,f\,dp \to 0, \quad \text{as} \quad h \to 0,\ h > 0;$$

(iii) for $f \in D(E)$,

$$\frac{2}{h^2}\,T(h)\int_0^h U(p)\,f\,dp \to Ef, \quad \text{as} \quad h \to 0,\ h > 0.$$

PROOF: (i) follows from the following two facts:

(a) $$\frac{1}{h^2}\left[\int_t^{t+h} (p-t)\,dp + \int_{t-h}^{t} (t-p)\,dp\right] = 1,$$

and (b) the regularity of $\{U(t)\}$.

(ii) If $f \in D(E)$, for any $\epsilon > 0$, there exists $h_0 > 0$ such that

$$\left\|\frac{U(p)\,f}{p} - Ef\right\| < \epsilon/3, \quad \text{for} \quad 0 < p \leqslant h_0$$

that is, $\|\,U(p)\,f - pEf\,\| < \frac{1}{3}\epsilon p$, for $0 \leqslant p < h_0$.

Hence for $0 \leqslant h \leqslant h_0$,

$$\left\|\frac{2}{h^2}\,T(h)\int_0^h U(p)\,f\,dp - Ef\right\| =$$

(*equation continued on p.* 598)

$$= \left\| \frac{2}{h^2} T(h) \int_0^h (U(p) f - pEf)\, dp + T(h) Ef - Ef \right\|$$

$$\leqslant \| T(h) \| (\tfrac{1}{3} \epsilon) + \| T(h) Ef - Ef \|.$$

By taking h sufficiently small, we see that

$$\left\| \frac{2}{h^2} T(h) \int_0^h U(p) f\, dp - Ef \right\| \text{ can be made as small as we please.}$$

Hence $$\frac{2}{h^2} \int_0^h U(p) f\, dp \to Ef \quad \text{as} \quad h \to 0, h > 0 \quad \text{for} \quad f \in D(E).$$

Let $\{T(t); t \in R^+\}$, $T : R^+ \to B(X)$, be a (C_0)-semigroup. Then the following two results hold (ref. Hille and Phillips 1957) :

(i) $$\frac{1}{h} \int_0^h T(p) f\, dp \to f, \quad \text{as} \quad h \to 0, h > 0, f \in X$$

and (ii) $$\frac{2}{h^2} \int_0^h pT(p) f\, dp \to f, \quad \text{as} \quad h \to 0, h > 0, f \in X.$$

The following corollary follows easily from the above result and Lemma 1:

Corollary 1 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then for $f \in D(E)$,

(i) $$\left(\frac{1}{h^2}\right) U(h) \int_0^h T(p) f\, dp \to Ef \quad \text{as} \quad h \to 0, h > 0,$$

(ii) $$\left(\frac{2}{h^2}\right) U(h) \int_0^h pT(p) f\, dp \to 0 \quad \text{as} \quad h \to 0, h > 0.$$

Lemma 2 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then for each $f \in D(E)$,

$$F_h \int_0^t (t - p)\, U(p) f\, dp \to U(t) f - tEf$$

as $h \to 0, 0 < h < t$.

PROOF : Consider

$$F_h \int_0^t (t-p)\, U(p) f\, dp$$

$$= \left(\frac{1}{h^2}\right) \int_0^h (t-p)(2U(p)\, T(h) f - 2U(p)\, U(h) f - 2U(p) f)\, dp$$

$$= \left(\frac{1}{h^2}\right) \int_0^h (t-p)\, (2U(p)\, T(h) f - 2U(h)\, U(p) f)\, dp$$

$$+ \left(\frac{1}{h^2}\right) \int_h^t (t-p)\, (2U(p)\, T(h) f - 2U(p)\, U(h) f)\, dp$$

$$- \left(\frac{2}{h^2}\right) \int_0^t (t-p)\, U(p) f\, dp$$

$$= \left(\frac{2}{h^2}\right)\Big((tT(h) \int_0^h U(p) f\, dp - T(h) \int_0^h pU(p) f\, dp)$$

$$- (tU(h) \int_0^h T(p) f\, dp - U(h) \int_0^h pT(p) f\, dp)\Big)$$

$$- \left(\frac{1}{h^2}\right)\left(\int_0^{t+h} (p-t)\, U(p) f\, dp + \int_{t-h}^t (t-p)\, U(p) f\, dp\right)$$

$$- \left(\frac{2}{h^2}\right) \int_0^h (h-p)\, U(p) f\, dp + \left(\frac{1}{h}\right) \int_{t-h}^{t+h} U(p) f\, dp, \text{ (using (1))},$$

$$= H_1 - H_2 - H_3 + H_4, \text{ say.}$$

By Lemma 1, $H_4 \to 2U(t) f$, $H_3 \to 0$ and $H_2 \to U(t) f$, as $h \to 0$, $h > 0$. Finally,

$$H_1 = t\left(\frac{2}{h^2}\right) T(h) \int_0^h U(p) f\, dp - \left(\frac{2}{h^2}\right) T(h) \int_0^h pU(p) f\, dp$$

$$- 2t\left(\frac{1}{h^2}\right) U(h) \int_0^h T(p) f\, dp + \left(\frac{2}{h^2}\right) U(h) \int_0^h pT(p) f\, dp$$

which, for $f \in D(E)$, converges to $tEf - 0 - 2tEf + 0 = -tEf$, by Lemma 1 and Corollary 1.

Hence

$$F_h \int_0^t (t-p)\, U(p) f\, dp \to -tEf - U(t) f - 0 + 2U(t) f$$

$$= U(t) f - tEf$$

as $h \to 0$, $h > 0$, for $f \in D(E)$.

Corollary 2 — Let $\{U(t);\, t \in R^+\}$, $U: R^+ \to B(X)$, be a regular modified sine operator. Then for each $f \in D(E)$,

$$\int_0^t (t-p)\, U(p) f\, dp \in D(F),$$

and $$F \int_0^t (t-p)\, U(p) f\, dp = U(t) f - tEf.$$

Lemma 3 — Let $\{U(t);\, t \in R^+\}$, $U: R^+ \to B(X)$, be a regular modified sine operator. Then for every $f \in D(F)$,

$$\int_0^t (t-p)\, U(p)\, Ff\, dp = U(t) f - tEf.$$

PROOF: If $f \in D(F)$, then, by the Proposition 1, $f \in D(E)$. By Lemma 2, for every $f \in D(E)$,

$$F_h \int_0^t (t-p)\, U(p) f\, dp \to U(t) f - tEf, \quad \text{as} \quad h \to 0,$$

$h > 0$. But

$$F_h \int_0^t (t-p)\, U(p) f\, dp = \int_0^t (t-p)\, U(p)\, F_h f\, dp.$$

Hence

$$\int_0^t (t-p)\, U(p)\, F_h f\, dp \to U(t) f - tEf \quad \text{as} \quad h \to 0,$$

for $f \in D(E)$. Therefore, for $f \in D(F)$,

$$\int_0^t (t-p)\, U(p)\, Ff\, dp = U(t) f - tEf.$$

Proposition 2 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. If $\bigcup_{p>0} X_p = X$, where $X_p = [U(p)]\, X$, then $D(F^r)$ is dense in X for each $r = 1, 2, 3, \ldots$; furthermore, $\bigcap_{r=1}^{\infty} D(F^r)$ is dense in X.

PROOF: Let $C_{00}^{\infty}(E_1^+)$ be the class of all numerically valued functions, defined on the positive real axis $E_1^+ = \{t; t > 0\}$, having continuous derivatives of all orders and compact support. If $\varphi \in C_{00}^{\infty}(E_1^+)$, so is its rth derivative $\varphi^{(r)} \in C_{00}^{\infty}(E_1^+)$, for all $r = 1, 2, 3, \ldots$, and the mapping

$$p \to \varphi(p)\, U(p) f$$

defines a strongly continuous vector valued function on E_1^+ to X, for each $f \in X$. Let X_{00} be the set of all elements of the form

$$g = \int_0^{\infty} \varphi(p)\, U(p) f\, dp,\ \varphi \in C_{00}^{\infty}(E_1^+),\ f \in X.$$

(The integral is well defined). X_{00} is a linear manifold in X. We shall show that $X_{00} \subset D(F^r)$ for each $r = 1, 2, 3, \ldots$, and that X_{00} is dense in X.

Now, for h sufficiently small,

$$F_h g = \left(\frac{2}{h^2}\right) \int_0^{\infty} \varphi(p)\, (T(h)\, U(p) f - U(h)\, U(p) f - U(p) f)\, dp$$

$$= \left(\frac{2}{h^2}\right) \int_0^{u} \varphi(p)\, (T(h)\, U(p) f - U(h)\, U(p) f - U(p) f)\, dp$$

$$+ \left(\frac{2}{h^2}\right) \int_u^{\infty} \varphi(p)\, (T(h)\, U(p) f - U(h)\, U(p) f - U(p) f)\, dp,$$

where $u > 0$ may be conveniently selected.

Now $\varphi(p)$ has a compact support in E_1^+, therefore, there exist a and b such that $\varphi(p) = 0$, for $p < a$ and $p > b$. Now choose u such that $0 < h < u < a$, and h so small that $u + h < a$.

Then

$$F_h g = \left(\frac{2}{h^2}\right) \int_u^\infty \varphi(p)\,(T(h)\,U(p)f - U(h)\,U(p)f - U(p)f)\,dp$$

$$= \left(\frac{1}{h^2}\right)\left(\int_{u+h}^\infty \varphi(p-h)\,U(p)f\,dp + \int_{u-h}^\infty \varphi(p+h)\,U(p)f\,dp \right.$$

$$\left. - 2\int_u^\infty \varphi(p)\,U(p)f\,dp\right)$$

$$= \left(\frac{1}{h^2}\right) \int_u^\infty (\varphi(p-h) + \varphi(p+h) - 2\varphi(p))\,U(p)f\,dp$$

$$= \int_u^\infty \left(\frac{1}{h^2}\right)(\varphi(p+h) - 2\varphi(p) + \varphi(p-h))\,U(p)f\,dp$$

which tends to $\int_u^\infty \varphi''(p)\,U(p)f\,dp = \int_0^\infty \varphi''(p)\,U(p)f\,dp$, $h \to 0$, as $h > 0$, because

$$\left(\frac{1}{h^2}\right)(\varphi(p+h) - 2\varphi(p) + \varphi(p-h))$$

tends to $\varphi''(p)$, as $h \to 0$, $h > 0$, uniformly with respect to p on the support of φ.

Thus

$$Fg \equiv s - \lim_{\substack{h \to 0,\\ h>0}} F_h g = \int_0^\infty \varphi''(p)\,U(p)f\,dp.$$

Repeating the arguments we have $g \in D(F^r)$ for each integer $r > 0$, and

$$F^r g = \int_0^\infty \varphi^{(2r)}(p)\,U(p)f\,dp$$

which shows that $X_{00} \subset \bigcap_{r=1}^\infty D(F^r)$.

By $\langle f^*, f \rangle$, $f^* \in X^*$, $f \in X$, we shall mean the value of the linear functional f^* acting on f. Let us suppose that X_{00} is dense in X. Then there exists an element $f_0 \in X$ having positive distance from X_{00}, and by a Corollary of the Theorem of Hahn

Banach (Butzer and Berens 1967) a bounded linear functional f_0^* on X such that $\langle f_0^*, g \rangle = 0$ for all $g \in X_{00}$, and $\langle f_0^*, f_0 \rangle = 1$. This implies that

$$\langle f_0^*, \int_0^\infty \varphi(p)\, U(p) f\, dp \rangle = \int_0^\infty \varphi(p) \langle f_0^*, U(p) f \rangle\, dp$$

$$= 0 \qquad \ldots(10)$$

for each $\varphi \in C_{00}^\infty(E_1^+)$ and each $f \in X$. By our assumption,

$$\overline{\bigcup_{p \geqslant 0} U(p)\, X} = \overline{\bigcup_{p \geqslant 0} X_p} = X.$$

Therefore, for a given $\epsilon > 0$ and $f_0 \in X$, there exists an $f_1 = U(q)\, f_2$, for some $q > 0$ and $f_2 \in X$ such that

$$| \langle f_0^*, f_0 \rangle - \langle f_0^*, f_1 \rangle | < \epsilon.$$

Clearly $\langle f_0^*, f_1 \rangle \neq 0$, that is, $\langle f_0^*, U(q)\, f_2 \rangle \neq 0$. But $\langle f_0^*, U(p)\, f_2 \rangle$ is a continuous function in $p \geqslant 0$ with $\langle f_0^*, U(q)\, f_2 \rangle \neq 0$, it is easy to see that there is a $\varphi \in C_{00}^\infty(E_1^+)$ such that $\int_0^\infty \varphi(p) \langle f_0^*, U(p)\, f_2 \rangle\, dp \neq 0$. This is a contradiction to (10). Therefore X_{00} and also $\bigcap_{r=1}^\infty D(F^r)$ are dense in X.

Corollary 3 — Let $\{U(t);\, t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. If $\overline{\bigcup_{p \geqslant 0} X_p} = X$, where $X_p = [U(p)]\, X$, then $D(E^2)$ is dense in X, and so is $D(E)$.

PROOF : Follows from Propositions 1 and 2.

Proposition 3 — Let $\{U(t);\, t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then, for every $f \in D(F)$,

(i) $U(\,.\,) f$ is twice differentiable;

(ii) $(dU(t)/dt) f = Ef + \int_0^t U(p)\, Ff\, dp,\ t \in R^+$, and

$(dU(t)/dt) f \to Ef$, as $t \to 0,\ t \in R^+$;

(iii) $(d^2U(t)/dt^2) f = U(t)\, Ef,\ t \in R^+$, and

$(d^2U(t)/dt^2) f \to 0$, as $t \to 0,\ t \in R^+$.

PROOF: By Lemma 3, for $f \in D(F)$,

$$U(t)f = tEf + \int_0^t (t-p)\, U(p)\, Ff\, dp.$$

Therefore,

$$((U(t+h) - U(t))/h)f$$

$$= Ef + \left(\frac{1}{h}\right)\left(\int_0^{t+h} (t+h-p)\, U(p)\, Ff\, dp - \int_0^t (t-p)\, U(p)\, Ff\, dp\right)$$

$$= Ef + \int_0^{t+h} (U(p)\, Ff\, dp + \left(\frac{1}{h}\right)\int_t^{t+h} (t-p)\, U(p)\, Ff\, dp.$$

Taking limits as $h \to 0$, $h > 0$, we have

$$(d^+U(t)/dt)f = Ef + \int_0^t U(p)\, Ff\, dp$$

since, in view of (2), for $f \in D(F)$,

$$\left\|\left(\frac{1}{h}\right)\int_t^{t+h} (t-p)\, U(p)\, Ff\, dp\right\| = \left\|\left(\frac{1}{h}\right)\int_0^h pU(t+p)\, Ff\, dp\right\|$$

$$\leqslant M \exp(\omega(t+h)) \left\|\left(\frac{1}{h}\right)\int_0^h pFf\, dp\right\|$$

$$\leqslant M \exp(\omega(t+h))\, \|Ff\| \left(\frac{h}{2}\right)$$

which converges to zero, as $h \to 0$, $h > 0$. Similarly,

$$d^-U(t)/dt)f = Ef + \int_0^t U(p)\, Ff\, dp, \; f \in D(F).$$

Now as $t \to 0$, $\int_0^t U(p)\, Ff\, dp \to 0$, for

$$\left\| \int_0^t U(p)\, Ff\, dp \right\| \leqslant M \exp(\omega t)\, \|Ff\|.$$

Hence $(dU(t)/dt)f \to Ef$, as $t \to 0$, for $f \in D(F)$, which establishes (ii).

Similarly, one can check that, for $f \in D(F)$, $(d^2U(t)/dt^2)f = U(t)\, Ff$, $t \in R^+$, which establishes (i) as well as a part of (iii). Taking limit as $t \to 0$, $t > 0$, and using the fact that $U(t)f \to 0$, as $t \to 0$, $t > 0$, we obtain $(d^2U(t)/dt^2)f \to 0$, as $t \to 0$, $t > 0$, $f \in D(F)$, which completes the proof of (iii).

Lemma 4 — Let $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then, for every $f \in D(F)$, $t \in R^+$,

$$\| U(t)f - tEf \| \leqslant (t^2/2) \| Ff \| \sup_{0 \leqslant p < t} \| U(p) \|.$$

PROOF : Using Lemma 3, we have for every $f \in D(F)$, $t \in R^+$,

$$U(t)f - tEf = \int_0^t (t - p)\, U(p)\, Ff\, dp.$$

Hence

$$\| U(t)f - tEf \| = \| \int_0^t (t - p)\, U(p)\, Ff\, dp \|$$

$$\leqslant (\int_0^t (t - p)\, dp) \| Ff \| \sup_{0 \leqslant p \leqslant t} \| U(p) \|$$

$$= (t^2/2) \| Ff \| \sup_{0 < p < t} \|U(p) \|$$

which proves the Lemma.

3. Representation Theorem

It is easy to see that, in the uniform operator topology, all the above results for the regular modified sine operator $\{U(t)\}$ will hold for all $f \in X$. In particular, from Lemma 2, it will follow that

$$U(t) = tE + E^2 \int_0^t (t - p)\, U(p)\, dp, \; t \in R^+, \qquad \text{...(11)}$$

where now E is a bounded linear operator defined on all of X, and $F = E^2$. Substituting for $U(p)$ on the right-hand side on (11), we will have

$$U(t) = tE + E^2 \int_0^t (t - p)\, (pE + E^2 \int_0^p (p - p_1)\, U(p_1)\, dp_1)\, dp$$

$$= tE + (t^3E^3/3!) + E^4 \int_0^t \int_0^p (t - p)\, (p - p_1)\, U(p_1)\, dp_1\, dp.$$

By induction

$$U(t) = \sum_{r=0}^{n-1} (t^{2r+1}/(2r + 1)!)\, E^{2r+1} +$$

(*equation continued on p. 606*)

$$+ E^{2n} \int_0^t \int_0^p \int_0^{p_1} \cdots \int_0^{p_{n-2}} (t - p)(p - p_1) \cdots (p_{n-2} - p_{n-1})$$

$$\times \; U(p_{n-1}) \, dp_{n-1} \, . \, dp_{n-2} \ldots dp_1 \ldots dp$$

for every positive integer n. Clearly the norm of the second term is bounded by $\| E \|^{2n} (M \exp(\omega t)) (t^{2n+1}/(2n + 1)!)$, which converges to 0, as $n \to \infty$, for a fixed $t \geqslant 0$, since $\| E \|$ is finite. Hence we have the following representation in the uniform operator topology :

$$U(t) = \sum_{r=0}^{\infty} (t^{2r+1}/(2r + 1)!) \, E^{2r+1}$$

$$= \sinh(tE), \; t \geqslant 0.$$

4. Examples

We had mentioned in an earlier paper some examples of the regular modified sine operator (see Ramesh Chander and Buche 1981). In this section we shall point out their infinitesimal generators and the associated differential equations.

(a) In Example 1 of Ramesh Chander and Buche (1981), $\{U(t); t \in R^+\}$ is defined on $C(R)$ as

$$[U(t) f](x) = (\tfrac{1}{2}) (f(x + at) - f(x - at)) \qquad \ldots(12)$$

where $a \neq 0$ is some constant, $f \in C(R)$ and $x \in R$, the real line.

The first and the second infinitesimal generators E and F of (12), are easily found to be

$$E = a(d/dx), \quad F = a^2(d^2/dx^2).$$

Hence the partial differential equation associated with (12), given by Proposition 3 is

$$(\partial u(x, t)/\partial t^2) - a^2(\partial^2 u(x, t)/\partial x^2) = 0, \qquad \ldots(13)$$

where $u(x, t) = [U(t) f](x)$ and $u(x, 0) = 0$,

$$(\partial u(x, t)/\partial t)_{t=0} = af'(x).$$

Here it is assumed that f has an absolutely continuous derivative.

(b) In the Example 2 of Ramesh Chander and Buche (1981) the modified sine operator $\{U(t)\}$ is defined on $C(R)$ as :

$$[U(t) f](x) = \sum_{k=0}^{\infty} (\alpha t)^{2k+1}/(2k + 1)!) \, f(x - (2k + 1) \, \mu)$$

where $\mu > 0$ is some constant (cf. Buche 1971).

The first and second infinitesimal generators of $\{U(t)\}$ defined by (14) are given by

$$Ef(x) = \alpha f(x - \mu),$$

and

$$Ff(x) = \alpha^2 f(x - 2\mu).$$

By Proposition 3, the associated differential equation is given by

$$(d^2u(t, x)/dt^2) - \alpha^2 u(t, x - 2\mu) = 0,$$

where $u(t, x) = [U(t) f] (x)$, $u(0, x) = 0$ and

$$(du(t, x)/dt)_{t=0} = \alpha f(x - \mu).$$

(c) In Example 3 of Ramesh Chander and Buche (1981), we have

$$Ef(x) = \sum_{-\infty}^{\infty} \lambda_n f_n e^{inx},$$

and

$$Ff(x) = \sum_{-\infty}^{\infty} \lambda_n^2 f_n e^{inx}.$$

(d) In Example 4 of Ramesh Chander and Buche (1981), we have $E = A$ and $F = A^2$.

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