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A CLASS OF TRIPLE EQUATIONS INVOLVING SERIES OF JACOBI AND LAGUERRE POLYNOMIALS

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The solution of a certain class of triple equations involving series of Jacobi polynomials has been obtained by reducing them to a Fredholm integral equation of second kind. This is a slightly more general problem than the one solved earlier by Srivastava and Panda (1978). Results for similar triple series equations involving Laguerre polynomials have also been obtained by applying a limit process.

1. INTRODUCTION

In this paper we consider certain triple series equations involving Jacobi polynomials, which are mild generalizations of those considered earlier by Srivastava and Panda (1978). We shall determine the sequence $\{A_n\}$ satisfying the triple series equations of the first kind.

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta + n + l + 1)}{\Gamma(\sigma + n + l + 1)} P_{n+l}^{(\lambda,\mu)}\left(1 - \frac{2x}{c}\right) = 0, \qquad \ldots(1.1)$$

$$0 < x < a, \quad b < x < c,$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma + n + l + 1)}{\Gamma(\rho + n + l + 1)} P_{n+l}^{(\alpha,\beta)}\left(1 - \frac{2x}{c}\right) = f(x), \quad a < x < b \qquad \ldots(1.2)$$

and the triple series equations of the second kind

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma + n + l + 1)}{\Gamma(\rho + n + l + 1)} P_{n+l}^{(\alpha,\beta)}\left(1 - \frac{2x}{c}\right) = g(x), \qquad \ldots(1.3)$$

$$0 < x < a, \quad b < x < c,$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta + n + l + 1)}{\Gamma(\rho + n + l + 1)} P_{n+l}^{(\lambda,\mu)}\left(1 - \frac{2x}{c}\right) = 0, \quad a < x < b \qquad \ldots(1.4)$$

where $c > 0$, l is an arbitrary nonnegative integer, $f(x)$ and $g(x)$ are prescribed functions and in general,

$$\min(\alpha, \beta, \nu, \delta, \lambda, \mu, \sigma, \rho) > -1. \qquad \ldots(1.5)$$

If $\rho = \alpha$, $\sigma = \mu$, $\alpha + \beta = \gamma + \delta = \lambda + \mu$ and $l = 0$ eqns. (1.1) to (1.4) reduce to those of equations considered by Dwivedi and Trivedi (1974). Furthermore, in limiting case

$$\lim_{c \to \infty} \left\{ P_n^{(\alpha,\beta)} \left(1 - \frac{2x}{c}\right) \right\} = L_n^{(\alpha)}(x) \qquad \ldots(1.6)$$

these equations would reduce to triple series equations involving generalized Laguerre polynomials studied by Dwivedi and Trivedi (1976).

In the analysis we shall use the following summation result:

$$S(r, x) =$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(\nu + n + l + 1)\,\Gamma(\sigma + n + l + 1)\,(n + l)!\,\Gamma(\lambda + \mu + n + l + 1)}{\Gamma(l + n + \rho + 1)\,\Gamma(\delta + n + l + 1)\,c^{\lambda+1}\,\Gamma(\lambda + n + l + 1)}$$

$$\times P_{n+l}^{(\alpha,\beta)}(1 - (2x/c))\,P_{n+l}^{(\lambda,\mu)}(1 - (2x/c))$$

$$= a_n^* \, x^{-\alpha} \, y^{-\lambda} \int_0^y \eta(t)\,(y - t)^{\mu'-1}\,(x - t)^{\alpha-\lambda+\mu'-1}\,dt,$$

$$= a_n^* \, x^{-\alpha} \, y^{-\lambda} \, S_y(r, x), \; y = \min(r, x) \qquad \ldots(1.7)$$

where

$$\eta(t) = t^{\lambda-\mu'}\,(1 - (t/c))^{-(\mu+\mu')} \qquad \ldots(1.8)$$

$$a_n^* = \frac{c^{2\alpha-2\lambda+\mu'}\,\Gamma(\gamma + n + l + 1)\,[\Gamma(\beta + n + l + 1)]^2\,\Gamma(\alpha - \lambda + \mu')}{\Gamma(\mu)\,\Gamma(\delta + n + l + 1)\,\Gamma(\lambda - \mu' + n + l + 1)}$$

$$\times \frac{\Gamma(\alpha - \lambda + \mu')\,\Gamma(\lambda + \mu + 2n + 2l + 1)}{\Gamma(\alpha + \beta - \lambda + \mu' + n + l + 1)\,\Gamma(\mu + n + l + 1)} \qquad \ldots(1.9)$$

it being assumed that the parameters are so constrained that a_n^* is independent of n. This is, of course, possible when for instance, $\alpha = \nu = \lambda - \mu'$, $\beta = \delta$.

2. Triple Series Equations of First Kind

We now turn to our series eqns. (1.1) and (1.2). If we assume eqn. (3.1) equal to an unknown function $\varphi(x)$ in (a, b), then on appealing to orthogonality relation given by Srivastava and Panda (1978), we get

$$A_n = \frac{\Gamma(\sigma + n + l + 1)\,(n + l)!\,(\lambda + \mu + 2n + 2l + 1)}{\Gamma(\delta + n + l + 1)\,c^{\lambda+1}\,\Gamma(\lambda + n + l + 1)} \times$$

(*equation continued on p.* 610)

$$\times \frac{\Gamma(\lambda + \mu + n + l + 1)}{\Gamma(\mu + n + l + 1)} \int_a^b x^\lambda (1 - x/c)^\mu \varphi(x)$$

$$\times P_{n+l}^{(\lambda,\mu)} (1 - (2x/c))\, dx. \qquad ...(2.1)$$

On substituting this A_n in (1.2) and interchanging the order of integration, we have

$$\int_a^x \eta(t) (x - t)^{\alpha-\lambda+\mu'-1} \bar{\varphi}(t)\, dt$$

$$= \frac{x^\alpha f(x)}{a_n^*} - \int_0^a \eta(r) (x - r)^{\alpha-\lambda+\mu'-1} Q(r)\, dr, \; a < x < b \qquad ...(2.2)$$

where

$$\bar{\varphi}(t) = \int_t^b \varphi(y) (1 - (y/c))^\mu (y - t)^{\mu'-1}\, dy, \; a < t < b \qquad ...(2.3)$$

$$Q(r) = \int_a^b \varphi(y) (1 - (y/c))^\mu (y - t)^{\mu'-1}\, dy, \; 0 < r < a \qquad ...(2.4)$$

and $\eta(t)$ is the same as defined earlier.

Inverting the Abel integral equations (2.2) and (2.4), we find

$$\eta(t)\bar{\varphi}(t) = F(t) - \frac{\sin(1 - \alpha + \lambda - \mu)\pi}{\pi} \int_0^a \frac{(a - r)^{\alpha-\lambda+\mu'}}{(t - r)}$$

$$\times \frac{\eta(r) Q(r)\, dr}{(t - a)^{\alpha-\lambda+\mu'}}, \quad a < t < b \qquad ...(2.5)$$

where

$$F(t) = \frac{\sin(1 - \alpha + \lambda + \mu')\pi}{\pi a_n^*} \frac{d}{dt} \int_a^t \frac{x^\alpha f(x)\, dx}{(t - x)^{\alpha-\lambda+\mu'}}. \qquad ...(2.6)$$

Also, after substituting for $\varphi(y)$ from (2.3) and integrating by parts and evaluating certain integrals, we get

$$Q(r) = \frac{\sin(1 - \mu')\pi}{\pi} (a - r)^{\mu'} \int_a^b \frac{\bar{\varphi}(t)\, dt}{(t - r)(t - a)^{\mu'}}. \qquad ...(2.7)$$

Thus from (2.5) and (2.7), we get

$$\eta(t)\bar{\varphi}(t) = F(t) - \int_a^b M(t, y) \bar{\varphi}(y)\, dy \qquad ...(2.8)$$

where

$$M(t, y) = \frac{\sin(1 - \alpha + \lambda - \mu')\pi \sin(1 - \mu')\pi}{\pi^2 (t - a)^{\alpha - \lambda + \mu'} (y - a)^{\mu'}}$$

$$\times \int_0^a \frac{(a - r)^{\alpha - \lambda + 2\mu'} \eta(r)}{(t - r)(y - r)} \, dr. \qquad \text{...(2.9)}$$

Equation (2.8) is a Fredholm integral equation of the second kind for $\bar{\varphi}(t)$. The coefficients A_n satisfying (1.1) and (1.2) can be found from eqns. (2.1) and (2.3).

3. Triple Series Equations of Second Kind

Next, we consider the equations (1.3) and (1.4). Then on proceeding as in section 2, we find that the solution of this set is given by the Fredholm equation

$$\eta(t) \bar{\varphi}_2(t) = T(t) - \int_b^c N(t, y) \bar{\varphi}_2(y) \, dy, \quad b < t < c \qquad \text{...(3.1)}$$

where $\eta(t)$ is the same as defined earlier and

$$T(t) = \frac{\sin(1 - \alpha + \lambda - \mu')}{\pi a_n^*} \frac{d}{dt} \int_b^t \frac{x^\alpha h(x) \, dx}{(t - x)^{\alpha - \lambda + \mu'}}$$

$$- \frac{\sin(1 - \alpha + \lambda - \mu')}{\pi (t - b)^{\alpha - \lambda + \mu'}} \int_0^a \frac{(b - r)^{\alpha - \lambda + \mu'} G(r) \, dr}{(t - r)} \qquad \text{...(3.2)}$$

$$N(t, y) = \frac{\sin(1 - \alpha + \lambda - \mu')\pi}{\pi^2 (t - b)^{\alpha - \lambda + \mu'}} \cdot \frac{\sin(1 - \mu')\pi}{(y - b)^{\mu'}}$$

$$\times \int_a^b \frac{(b - r)^{\alpha - \lambda + 2\mu'}}{(t - r)(y - r)} \eta(r) \, dr. \qquad \text{...(3.3)}$$

Once $\bar{\varphi}_2$ is determined from (3.1), φ_1, φ_2 are found from the following equations:

$$(1 - y/c)^\mu \varphi_2(y) = - \frac{\sin(1 - \mu')\pi}{\pi} \frac{d}{dy} \int_y^c \frac{\bar{\varphi}_2(t) \, dt}{(t - y)^{\mu'}}, \quad b < y < c, \qquad \text{...(3.4)}$$

$$R(r) = \frac{\sin(1 - \mu')\pi}{\pi} (b - r)^{\mu'} \int_b^c \frac{\varphi_2(t) \, dt}{(t - r)(t - b)^{\mu}} \qquad \text{...(3.5)}$$

$$\eta(t) \bar{\varphi}_1(t) = G(t) - \eta(t) R(t), \quad 0 < t < a \qquad \text{...(3.6)}$$

$$(1 - (y/c))^{\mu} \varphi_1(y) = - \frac{\sin (1 - \mu') \pi}{\pi} \frac{d}{dy} \int_y^a \frac{\bar{\varphi}_1(t)\, dt}{(t - y)^{\mu'}}, \quad 0 < y < a. \qquad ...(3.7)$$

Finally, the coefficients A_n are given by

$$A_n = \frac{\Gamma(\sigma + n + l + 1)\,(n + l)\,!\,\Gamma(\lambda + \mu + 2n + 2l + 1)}{\Gamma(\delta + n + l + 1)\, c^{\lambda+1}\, \Gamma(\lambda + n + l + 1)}$$

$$\times \frac{\Gamma(\lambda + \mu + n + 1)}{\Gamma(\mu + n + l + 1)} \left[\left\{ \int_0^a \varphi_1(y) + \int_b^c \varphi_2(y) \right\} y^{\lambda}(1 - (y/c))^{\mu} \right.$$

$$\left. \times P_{n+l}^{(\lambda,\mu)} (1 - (2y/c))\, dy \right]. \qquad ...(3.8)$$

4. Special and Limiting Cases

(i) For $\rho = \alpha$, $\sigma = \mu$, $\alpha + \beta = \nu + \delta = \lambda + \mu$, $l = 0$, the triple series equations (1.1) to (1.4) would reduce to those considered earlier by Dwivedi and Trivedi (1976) and indeed our solution with $l = 0$ can be shown fairly easily to be in complete agreement with the Dwivedi-Trivedi solution.

(ii) If $\rho = \alpha$, $\sigma = \mu$ and we replace β and μ in terms of δ from $\alpha + \beta = \nu + \delta = \lambda + \mu$ and set $\delta = c$, $A_n = B_n/\Gamma(\gamma + n + l + 1)$ then on taking the limits as $c \to \infty$, eqns. (1.1) and (1.2) reduce to

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\gamma + n + l + 1)} L_{n+l}^{(\lambda)}(x) = 0, \quad 0 < x < a, \quad b < x < \infty \qquad ...(4.1)$$

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\alpha + n + l + 1)} L_{n+l}^{(\alpha)}(x) = f(x), \quad a < x < b \qquad ...(4.2)$$

whose solution can be written down immediately from eqns. (2.8), (2.3) and (2.1). The solution so obtained would agree to the solution obtained by Dwivedi and Trivedi (1974).

Similarly, eqns. (1.3) and (1.4) reduce to

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\alpha + n + l + 1)} L_{n+l}^{(\alpha)}(x) = g(x), \quad 0 < x < a$$
$$= h(x), \quad b < x < \infty \qquad ...(4.3)$$

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\gamma + n + l + 1)} L_{n+l}^{(\lambda)}(x) = 0, \quad a < x < b \qquad \ldots(4.4)$$

whose solution agrees with that of Dwivedi and Trivedi (1974).

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