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ALMOST PRODUCT STRUCTURES ON TANGENT BUNDLE

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In the present paper the authors study the almost product structures on the tangent bundle $T(M)$ with the help of a Finsler tensor field of (1, 1)-type on the total space F of the Finsler bundle $F(M)$ of a differentiable manifold M.

1. INTRODUCTION

Let M be an n-dimensional differentiable manifold and let $L(M) = (L, \pi_L, M, GL(n, R))$ be the bundle of linear frame over M where $GL(n, R)$ is the general linear group, $T(M) = (T, \pi_T, M, V, GL(n, R))$ be the tangent bundle over M where V is an n-dimensional real vector space called standard fibre and $F(M) = (F, \pi_1, T, GL(n, R))$ be the Finsler bundle over M. The map $\pi_2 : F \to L$, $u = (y, z) \to z$ is called induced map and the map $l_y : M_x \to N_y$, $x = \pi_T(y)$ is called the lift with respect to a non-linear connection N to the point $y \in T$. Moreover $\pi_T l_y =$ identity.

Let (Γ, N) be a Finsler connection, (Γ^h, Γ^v) be the Finsler pair and (Γ_V, N, Γ^v) be the Finsler traid on a differentiable manifold M then from Matsumoto (1970)

$$\Gamma^h_u = l_u N_y, \quad \Gamma^v_u = l_u T^v_y$$

where T^v_y be the vertical subspace of the tangent space T_y.

Finsler homomorphism — Let $F(M) = (F, \pi_1, T, GL(n, R))$ be the Finsler bundle over a manifold M and let $L(T) = \bar{L}(M) = (\bar{L}, T, \bar{\pi}, \bar{G})$ be the bundle of linear frame over the total space T of the tangent bundle $T(M)$ where $\bar{G}$ is the general linear group $GL(2n, R)$. There exists a group homomorphism $b : G \to \bar{G}$ such that for $g : (g^a_b) \in G$, $b(g)$ is given by $\begin{pmatrix} g^a_b, & 0 \\ 0, & g^a_b \end{pmatrix}$. This b is called natural homomorphism.

Let N be the nonlinear connection on M, the Finsler homomorphism β_N is a bundle homomorphism $\beta_N : F \to \bar{L}$, defined by

$$\beta_N(u) = (l_y z, l^v_y z), \quad u = (y, z) \in F$$

where $l_y z = (l_y(z_a))$ is the lift of the linear frame $z = z_a$ $(a = 1, \ldots, n)$ to the point $y \in T$ with respect to given N and $l_y^v z = (l_y^v (z_a))$ is the vertical lift of z.

Let us consider the tangent bundle $T(T(M)) = \bar{T}(M) \equiv (\bar{T}, T, \bar{\pi}_T, \bar{V}, \bar{G})$ over the total space T of the tangent bundle $T(M)$. The standard fibre $\bar{V}$ is real $2n$-vector space can be identified with the product $V \times V$ of real n-space V which is the standard fibre of $T(M)$. Let (e_a) be the base of V, from this base we obtain the base $(\bar{e}_\alpha) = (\bar{e}_a, \bar{e}_{(a)})$, $\alpha = 1, \ldots, 2n$ of $\bar{V}$ such that $\bar{e}_a = (e_a, 0)$, $\bar{e}_{(a)} = (0, e_a)$, $(a) = n + a$.

Let $Z(A)$ be the fundamental vector field and $B^h(v)$ and $B^v(v)$ are h and v-basic vector field on F the total space of finsler bundle then

$$\left.\begin{aligned} \beta'_N Z(A) &= \bar{Z}(\bar{A}) = \bar{Z}\begin{pmatrix} A, & 0 \\ 0, & A \end{pmatrix}, \\ \beta'_N B^h(v) &= \bar{B}(v, 0), \quad \beta'_N B^v(v) = \bar{B}(0, v) \end{aligned}\right\} \qquad \ldots(1.1)$$

where $\bar{Z}(\bar{A})$ be the fundamental vector field on $\bar{L}(M)$, corresponding to $\bar{A} \in L(\bar{G})$ and $\bar{B}(0, v)$ and $\bar{B}(v, 0)$ are basic vector fields on $\bar{L}(M)$ and β'_N be the differential map of β_N.

N-decompositions of tensors — Let K be a tensor field of (1, 1)-type on T, it is regarded as a $\bar{V}_1^1$ valued function on $\bar{L}(T)$ which satisfies $K\bar{\tau}_{\bar{g}} = \bar{g}^{-1}K$, where $\bar{\tau}_{\bar{g}}$ is the right translation on $\bar{L}$ by $\bar{g} \in \bar{G}$. From K we deduce four mapings K_1^1, K_2^1, K_1^2, $K_2^2 : F \to V_1^1$ as follows:

$$\left.\begin{aligned} (K_1^1)_u (v^*, v) &= K_{\beta_N(u)} ((v^*, 0), (v, 0)) \\ (K_2^1)_u (v^*, v) &= K_{\beta_N(u)} ((v^*, 0), (0, v)) \\ (K_1^2)_u (v^*, v) &= K_{\beta_N(u)} ((0, v^*), (v, 0)) \\ (K_2^2)_u (v^*, v) &= K_{\beta_N(u)} ((0, v^*), (0, v)) \end{aligned}\right\} \qquad \ldots(1.2)$$

where $v^* \in V^*$, $v \in V$. These K_β^α $(\alpha, \beta = 1, 2)$ are Finsler tensor field of (1, 1)-type and V^* be the dual space of V.

These Finsler tensor fields K_β^α $(\alpha, \beta = 1, 2)$ as defined above are called N-decomposition of a tensor field K with respect to a nonlinear connection N. We have from Matsumoto (1970, 1977)

$$\left.\begin{aligned} K(v, 0)\, \beta_N &= (K_1^1(v), K_1^2(v)) \in \bar{V} \\ K(0, v)\, \beta_N &= (K_2^1(v), K_2^2(v)) \in \bar{V}. \end{aligned}\right\} \qquad \ldots(1.3)$$

Let X be a tangent vector field on T, let us construct the mappings X^- and $X^\perp : F \to V$ as follows:

$$X^{=}(u) = \theta^h l_u X, \; X^{\perp}(u) = \theta^v l_u X \qquad ...(1.4)$$

at a point $u \in F$, $X^{=}$ and $X^{\perp}$ are called Finsler h and v vector fields derived from X respectively.

Given a tangent vector field **X** on M, we obtain horizontal lift $\mathbf{X}^h$ and the vertical lift $\mathbf{X}^v$ with respect to non-linear connection N as follow

$$\mathbf{X}^h_y = l_y \mathbf{X}, \quad \mathbf{X}^v_y = z^{\alpha'} S_{\epsilon(u)}(v), \quad v = z^{-1}\mathbf{X} \qquad ...(1.5)$$

where $z^{\alpha} : V \rightarrow T$ is the admissible mapping and ϵ is the element of support and $S(v)$ be the parallel vector field on V. Then we have the results from Matsumoto (1966)

$$(\mathbf{X}^h)^{\perp} = 0, \; (\mathbf{X}^v)^{=} = 0$$

$$(\mathbf{X}^h)^{=} = (\mathbf{X}^v)^{\perp} = z^{-1}\mathbf{X}.$$

2. Almost Product Structure on the Tangent Bundle $T(M)$

Let us define a Finsler tensor field J of (1,1)-type on F, the total space of the Finsler bundle $F(M)$ such that

$$J(Z(A)) = Z(A), \quad J(B^h)(v)) = -B^v(v), \quad J(B^v(v)) = -B^h(v) \qquad ...(2.1)$$

where $B^h(v)$ and $B^v(v)$ are h and v basic vector fields on F and $Z(A)$ is the fundamental vector field on F respectively.

With respect to a Finsler connection $F\Gamma$ an arbitrary tangent vector $X \in F_u$ is decomposed as

$$X = Z(\omega X) + B^h(\theta^h X) + B^v(\theta^v X) \qquad ...(2.2)$$

where ω is the connection form and θ^h and θ^v are h and v basic forms on F.

Proposition 2.1 — We have

$$\omega J = \omega, \quad \theta^h J = -\theta^v, \quad \theta^v J = -\theta^h \qquad ...(2.3)$$

$$J \,.\, \tau_g = \tau_g \,.\, J \qquad ...(2.4)$$

where τ_g is the right translation on F by $g \in GL(n, R)$.

Proof is obvious from (2.1).

From (2.1) and (2.2) we have for an arbitrary tangent vector $X \in F_u$

$$J^2 X = X. \qquad ...(2.5)$$

The structure J is called an almost product structure on the Finsler bundle $F(M)$ (see Sinha and Yadava 1979). From the Finsler (1, 1) tensor field J on F we deduce a (1, 1) tensor field $\bar{J}$ on T the total space of the tangent bundle $T(M)$ as follows.

Define

$$\bar{J} = \pi_1 J l \qquad \text{...(2.6)}$$

where l is the operation of lift with respect to the Finsler connection under consideration. From (2.5) and (2.6) we have given any tangent vector $X \in T_y$, $y = \pi_1(u)$

$$\bar{J}^2 X = X. \qquad \text{...(2.7)}$$

Definition 2.1 — Let J be the almost product structure on the Finsler bundle $F(M)$. A (1, 1) tensor field $\bar{J}$ on T induced from J given by (2.6) is said to define an almost product structure on the tangent bundle $T(M)$.

Propositon 2.2 — If J is an almost product structure on $F(M)$ and $\bar{J}$ the almost product structure on $T(M)$, then

$$\bar{J}\pi_1 = \pi_1 J, \; Jl = l\bar{J} \qquad \text{...(2.8)}$$

$$\left.\begin{aligned} &\beta'_N J(Z(A)) = \bar{Z}\begin{pmatrix} A, & 0 \\ 0, & A \end{pmatrix}, \quad \beta'_N J(B^h(v)) = -\bar{B}(0, v) \\ &\beta'_N J(B^v(v)) = -\bar{B}(v, 0). \end{aligned}\right\} \qquad \text{...(2.9)}$$

PROOF : The proof of (2.8) is obvious from (2.6) and eqns. (2.9) follow from (2.1) and (1.1).

Proposition 2.3 — Let X be an arbitrary vector field on T and $\mathbf{X}$ be an arbitrary vector field on M. Then

$$(\bar{J}X)^{=} = -X^{\perp}, \quad (\bar{J}X)^{\perp} = -X^{=} \qquad \text{...(2.10)}$$

$$\bar{J}\mathbf{X}^h = -\mathbf{X}^v, \quad \bar{J}\mathbf{X}^v = -\mathbf{X}^h. \qquad \text{...(2.11)}$$

PROOF : From (1.4), (2.8) and (2.3) we have the first part of the proposition. Again using (1.5) and (2.6) with the definition of basic vector fields we get second part of the proposition.

We shall also define this almost product structure $\bar{J}$ on $T(M)$ by means of N-decompositions of tensors.

Definition 2.2 — Let N be a nonlinear connection in the tangent bundle $T(M)$. The almost product structure $\bar{J}$ on $T(M)$ is a tensor field of (1, 1)-type on T such that the N-decomposition $\bar{J}^{\alpha}_{\beta}$ ($\alpha, \beta = 1, 2$) of $\bar{J}$ are given by

$$\bar{J}^1_1 = \bar{J}^2_2 = 0, \; \bar{J}^2_1 = \bar{J}^1_2 = -\delta \qquad \text{...(2.12)}$$

where δ is Kronecker delta.

Let $v = (v_1, v_2) \in \bar{V} = V \times V$

then

$$\bar{J}(v) = \bar{J}(v_1, v_2) = \bar{J}(v_1, 0) + \bar{J}(0, v_2)$$
$$= (\bar{J}^1_1(v_1), \bar{J}^2_1(v_1)) + (\bar{J}^1_2(v_2), \bar{J}^2_2(v_2))$$
$$= (0, -v_1) + (-v_2, 0) = (-v_2, -v_1).$$

From which we deduce

$$\bar{J}(\bar{J}(v_1, v_2)) = (v_1, v_2).$$

Thus we are able to define the almost product structure $\bar{J}$ on $T(M)$ as follows.

Definition 2.3 — The almost product structure $\bar{J}$ on the tangent bundle $T(M)$ with respect to a nonlinear connection N is a (1, 1) tensor field : $\bar{L} \to \bar{V}^1_1$, defined by

$$\bar{J}((v_1, v_2)) = (-v_2, -v_1). \qquad \text{...(2.13)}$$

Proposition 2.4 — The above two Definitions 2.2 and 2.3 of the almost product structure $\bar{J}$ are equivalent.

PROOF : Using (2.13) we have

$$\bar{J}((v_1, 0)) = (0, -v_1) = (\bar{J}^1_1(v_1), \bar{J}^2_1(v_1))$$

$$\bar{J}((0, v_2)) = (-v_2, 0) = (\bar{J}^1_2(v_2), \bar{J}^2_2(v_2)).$$

From the above two equations we deduce

$$\bar{J}^1_1 = \bar{J}^2_2 = 0, \quad \bar{J}^2_1 = \bar{J}^1_2 = -\delta.$$

Remark 1 : Definitions 2.2 and 2.3 of almost product structure $\bar{J}$ on $T(M)$ show that $\bar{J}$ is really defined by means of a nonlinear connection N. Thus it is more proper to recall the almost product structure $\bar{J}$ as the almost product N-structure.

3. Almost Product N-structure on Tangent Bundle $T(M)$

Let G be a Finsler metric tensor. We define a tensor field $\bar{G}$ of (0, 2)–type on T as follows

$$\bar{G}(X_v, Y_v) = G(X^-(u), Y^-(u)) + G(X^\perp(u), Y^\perp(u)) \qquad \text{...(3.1)}$$

for the tangent vector fields X and Y on T at the point $y \in T$, $y = \pi_1(u)$.

Now $\bar{G}(X_v, X_v)^{1/2}$ defines the length of X, we have a Riemannian metric $\bar{G}$ on T and thus the tangent bundle $T(M)$ is a Riemannian manifold.

The tensor field $\bar{G}$ as thus defined is called lifted Riemannian metric on the tangent bundle $T(M)$ over M.

Proposition 3.1 — Let $\bar{G}$ be the lifted Riemannian metric on $T(M)$ derived from the Finsler metric G. If $\bar{J}$ is the almost product N-structure on $T(M)$, then $(\bar{J}, \bar{G})$ is an almost product metric N-structure on $T(M)$.

PROOF : From (3.1) we have

$$\bar{G}\,(\bar{J}X, \bar{J}Y) = G((\bar{J}X)^{=}, (\bar{J}Y)^{=}) + G((\bar{J}X)^{\perp}, (\bar{J}Y)^{\perp})$$

$$= G(-X^{\perp} - Y^{\perp}) + G(-X^{=}, -Y^{=}) = \bar{G}(X, Y). \quad \text{...(3.2)}$$

This completes the proof.

Proposition 3.2 — The almost product metric N-structure $(\bar{J}, \bar{G})$ on $T(M)$ is not unique.

PROOF : Define for $X, Y \in T_v$

$$\mu(J'(X)) = \bar{J}(\mu(X))$$

$$G'(X, Y) = \bar{G}(\mu(X), \mu(Y))$$

where μ is a non-singular tensor field of (1, 1)-type on T. It is easy to see that (J', G') will also define an almost product N-structure on $T(M)$.

Let us define

$$*J(X, Y) = \bar{G}(\bar{J}X, Y) \quad \text{...(3.3)}$$

for the tangent vectors $X, Y \in T_v$. We observe that $*J$ is a 2-form on T. Then we have the following proposition.

Proposition 3.3 — If $(\bar{J}, \bar{G})$ defines an almost product metric N-structure on $T(M)$, then

$$\bar{G}(\bar{J}X, Y) = \bar{G}(X, \bar{J}Y) \quad \text{...(3.4)}$$

$*J$ is pure. ...(3.5)

PROOF : From (3.1) we have

$$\bar{G}(\bar{J}X, Y) = \bar{G}((\bar{J}X)^{=}, Y^{=}) + \bar{G}((\bar{J}X)^{\perp}, Y^{\perp}))$$

$$= G(-X^{\perp}, Y^{=}) + G(-X^{=}, Y^{\perp})$$

$$\bar{G}(X, \bar{J}Y) = G(X^{-}, (\bar{J}Y)^{-}) + G(X^{\perp}, (\bar{J}Y)^{\perp})$$
$$= G(X^{-}, -Y^{\perp}) + G(X^{\perp}, -Y^{-}).$$

Thus we have (3.4).

Now from (3.3)

$$^{*}J(\bar{J}X, \bar{J}Y) = \bar{G}(\bar{J}^2X, \bar{J}Y) = \bar{G}(X, \bar{J}Y)$$
$$= \bar{G}(\bar{J}X, Y) = {}^{*}J(X, Y).$$

4. Lifted Nijenhuis Tensor

Let N be the Nijenhuis tensor of the almost product structure on the Finsler bundle $F(M)$. Then for tangent vector fields X, Y on F

$$N(X, Y) = [JX, JY] + [X, Y] - J[JX, Y] - J[X, JY]. \quad ...(4.1)$$

Let us define a tensor field $\bar{N}$ of (1, 2)-type on T as follows:

$$\bar{N}(X, Y) = N(X^{-}, Y^{\perp}) + N(X^{\perp}, Y^{-}) \quad ...(4.2)$$

for the tangent vector fields X, Y on T.

Definition 4.1 — The tensor field $\bar{N}$ of (1, 2)-type on T induced from the Nijenhuis tensor N of J given by (4.2) is called lifted Nijenhuis tensor of the almost product metric N-structure $(\bar{J}, \bar{G})$.

Proposition 4.1 — If $\bar{N}$ is the lifted Nijenhuis tensor of $(\bar{J}, \bar{G})$, then

$$\bar{N}(\mathbf{X}^h, \mathbf{Y}^h) = 0, \bar{N}(\mathbf{X}^v, \mathbf{Y}^v) = 0 \quad ...(4.3a)$$

$$\bar{N}(\mathbf{X}^h, \mathbf{Y}^v) = \bar{N}(\mathbf{X}^v, \mathbf{Y}^h) \quad ...(4.3b)$$

where $\mathbf{X}$, $\mathbf{Y}$ are tangent vector fields on M.

Proof is obvious from (4.2) and (2.11).

Proposition 4.2 — If $\bar{N}$ is the lifted Nijenhuis tensor of the almost product metric N-structure $(\bar{J}, \bar{G})$, then

$$\bar{N}(\bar{J}X, \bar{J}Y) = \bar{N}(X, Y) \quad ...(4.4a)$$

$$\bar{N}(\bar{J}XY) = \bar{N}(X, \bar{J}Y). \quad ...(4.4b)$$

Proof is obvious.

5. Condition of Integrability of Almost Product N-structure on $T(M)$

We already know that the almost product structure $(\bar{J}, \bar{G})$ on the tangent bundle $T(M)$ is integrable if the Nijenhuis tensor $^*\bar{N}$ of $\bar{J}$ vanishes.

$$^*\bar{N}(X, Y) = [\bar{J}X, \bar{J}Y] + [X, Y] - \bar{J}[\bar{J}X, Y] - \bar{J}[X, \bar{J}Y] \qquad ...(5.1)$$

For the Nijenhuis tensor $^*\bar{N}$ we have the following proposition.

Proposition 5.1 — For the Nijenhuis tensor $^*\bar{N}$ of an almost product metric N-structure $(\bar{J}, \bar{G})$ on $T(M)$, we have for the tangent vector fields **X**, **Y** on M.

$$^*\bar{N}(\mathbf{X}^h, \mathbf{Y}^h) = {}^*\bar{N}(\mathbf{X}^v, \mathbf{Y}^v) = \bar{J}\, {}^*\bar{N}(\mathbf{X}^h, \mathbf{Y}^v) = \bar{J}\, {}^*\bar{N}(\mathbf{X}^v, \mathbf{Y}^h) \qquad ...(5.2)$$

Proof : Using (5.1) and (2.11) we have the proof.

In terms of the notations and terms used in Yano and Ishihara (1967) we have given a tensor field $R = R^{i_1, i_2, \ldots, i_r}_{i, j_1 j_2, \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^i \otimes \ldots \otimes dx^{i_s}$ we define a tensor field rR on T, the total space of the tangent bundle (TM), by

$$rR = y^i R^{i_1, i_2, \ldots i_r}_{i, j_1, j_2, \ldots, j_s} \frac{\partial}{\partial y^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

we introduce an affine connection $\hat{\nabla}$ in M by

$$\hat{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} - S(\mathbf{X}, \mathbf{Y})$$

where **X**, **Y** being arbitrary vector fields on M and S be the torsion tensor of the given affine connection ∇. Then we have the following from Yano and Ishihara (1967) :

$$\left.\begin{aligned} [\mathbf{X}^h, \mathbf{Y}^h] &= [\mathbf{X}, \mathbf{Y}]^h - r\hat{R}(\mathbf{X}, \mathbf{Y}) \\ [\mathbf{X}^v, \mathbf{Y}^h] &= [\mathbf{X}, \mathbf{Y}]^h - (\nabla_{\mathbf{X}}\mathbf{Y})^v \\ [\mathbf{X}^v, \mathbf{Y}^v] &= 0 \end{aligned}\right\} \qquad ...(5.3)$$

where $\hat{R}$ be the curvature tensor of the affine connection $\hat{\nabla}$. Then we have the following theorem.

Theorem 5.2 — The condition that the almost product metric N-structure $(\bar{J}, \bar{G})$ be integrable is

$$r\hat{R}(\mathbf{X}\mathbf{Y}) - \bar{J}(r\hat{R}(\mathbf{X}, \mathbf{Y})) = S(\mathbf{X}, \mathbf{Y}))^v + (S(\mathbf{X}, \mathbf{Y})^h. \qquad ...(5.4)$$

Proof : From (5.3) and (5.2) and using (5.1) we have

$$^*\bar{N}(\mathbf{X}^h, \mathbf{Y}^h) = 2[\mathbf{X}, \mathbf{Y}]^h - r\hat{R}(\mathbf{X}, \mathbf{Y}) - 2[\mathbf{X}, \mathbf{Y}]^v + (S(\mathbf{X}, \mathbf{Y}))^h \qquad ...(5.5)$$

$$*\bar{N}(X^v, Y^h) = 2[X, Y]^h - 2[X, Y]^v - J(r\hat{R}(X, Y)) - (S(X, Y))^v. \quad ...(5.6)$$

The above two equations complete the proof of the theorem.

Let N be the Nijenhuis tensor of the almost product structure J of the finsler bundle $F(M)$ given by (4.1). Then from Sinha and Yadava (1979)

$$\left.\begin{aligned} N(B^h(v_1), B^h(v_2)) &= B^h(L(v_1, v_2)) + Z(M(v_1, v_2)) + B^v(R^1(v_1, v_2)) \\ N(B^h(v_1), B^v(v_2)) &= Z(M(v_1, v_2)) - B^h(R^1(v_1, v_2)) - B^v(L(v_1, v_2)) \\ N(B^v(v_1), B^v(v_2)) &= B^h(L(v_1, v_2))) + Z(M(v_1, v_2)) + B^v(R^1(v_1, v_2)) \end{aligned}\right\} \quad ...(5.7)$$

where

$$M(v_1, v_2) = R^2(v_1, v_2) - P^2(v_2, v_1) + P^2(v_1, v_2) + S^2(v_1, v_2)$$

$$L(v_1, v_2) = T(v_1, v_2) + P^1(v_2, v_1) - P^1(v_1, v_2).$$

We have the following proposition.

Proposition 5.3 — If N be the Nijenhuis tensor of J and $*N$ be the Nijenhuis tensor of $\bar{J}$, then

$$\pi_1 N(l_u X, l_u Y) = *\bar{N}(X, Y) \quad ...(5.8)$$

where

$$X, Y \in T_y, y = \pi_1(u).$$

PROOF : Using (2.8) and (4.1) we have

$$\pi_1 N(l_u X, l_u Y) = \pi_1[l_u \bar{J}X, l_u \bar{J}Y] + \pi_1[l_u X, l_u Y]$$
$$- \bar{J}\pi_1[l_u JX, l_u Y] - \bar{J}\pi_1[l_u X, l_u \bar{J}Y]$$

It will be known from Nomizu (1956) that the horizontal part of $[l_u X, l_u Y]$ is equal to the lift of $[X, Y]$ and hence $\pi_1[l_u X, l_u Y] = [X, Y]$ ($\pi_1 l_u =$ identity). Therefore, we obtain (5.12) with the help of (5.1).

Theorem 5.4 — The condition of integrability of the almost product metric N-structure $(\bar{J}, \bar{G})$ on $T(M)$ is that the tensor R^1 and L vanish identically.

PROOF : From (5.7) and using (5.8) we have

$$*\bar{N}(\pi_1 B^h(v_1), \pi_1 B^h(v_2)) = \pi_1 B^h(L(v_1, v_2)) + \pi_1 B^v(R^1(v_1, v_2))$$

$$*\bar{N}(\pi_1 B^h(v_1), \pi_1 B^v(v_2)) = - \pi_1 B^h(R^1(v_1, v_2)) - \pi_1 B^v(L(v_1, v_2))$$

$$*\bar{N}(\pi_1 B^v(v_1), \pi_1 B^v(v_2)) = \pi_1 B^h(L(v_1, v_2)) + \pi_1 B^v(R^1(v_1, v_2)).$$

Since $\pi_1 l_u =$ identity and $\pi_1 Z(A) = 0$.

From these above equations we conclude the proof.

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