Indian J. pure appl. Math., **12**(5) : 614–620, May 1981

RECURRENCE RELATIONS FOR THE GENERALIZED POLYNOMIAL SET

R. B. SINGH

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005

(*Received* 14 *April* 1980; *after revision* 25 *July* 1980)

The present note aims at deriving a few recurrence relations for the generalized polynomial set $\{\bar{R}_n(x, y)\}$ of order n, defined in Singh and Pandey (1977).

1. INTRODUCTION

Recently, Singh and Pandey (1977) have introduced a generalized polynomial set $\{\bar{R}_n(x, y)\}$ by means of the generating function

$$\sum_{n=0}^{\infty} \bar{R}^{m;m_1;m_2;m_3;m_4;\alpha;\mu;(a_p,A_p);l_2}_{n;\beta;v;\alpha_1;\nu_1;(b_{q+1},B_{q+1});r_1;l_1}(x, y)\, t^n = QJ_{\nu_1}\frac{(\alpha_1 x^{m_1} y^{m_1} t^{m_3})}{(1 - vx^{-m}t^{m_4})^{\alpha}}$$

$$\times H^{l_1,l_2}_{p,q+1}\left[\frac{-\mu y^{r_1} t}{(1 - vx^{-m}t^{m_4})^{\beta}}\middle|\begin{matrix}(a_1,A_1), (a_2, A_2), \ldots, (a_p, A_p)\\ (b_1, 1), (b_2, B_2), \ldots, (b_{q+1}, B_{q+1})\end{matrix}\right] \quad \ldots(1.1)$$

where $\mu \neq 0$, $v \neq 0$, $\alpha \geqslant 0$ m and m_i $(i = 1, 2, 3, 4)$ are positive integers,

$$\sum_{j=1}^{p} A_j - \sum_{j=2}^{q+1} B_j \leqslant 1$$

$$\sum_{j=1}^{l_2} A_j - \sum_{j=l_2+1}^{P} A_j + \sum_{j=2}^{l_1} B_j - \sum_{j=l_1+1}^{q+1} B_j \equiv \lambda' > 0$$

$$\left|\arg\frac{\mu y^{r_1} t}{(1 - vx^{-m}t^{m_4})^{\beta}}\right| < \tfrac{1}{2}\pi\lambda'$$

and

$$Q = \frac{\prod_{j=l_2+1}^{p}\Gamma(a_j - A_j b_1)\prod_{j=l_1+1}^{q+1}\Gamma(1 - b_j + B_j b_1)\, 2^{\nu_1}\, \nu_1!}{\prod_{j=1}^{l_2}\Gamma(1 - a_j + A_j b_1)\prod_{j=2}^{l_1}\Gamma(b_j - B_j b_1)\,(-\mu y^{r_1} t)^{b_1}\,(\alpha_1 x^{m_1} y^{m_2} t^{m_3})^{\nu_1}}.$$

The right-hand side of (1.1) contains the H-function due to Fox (1961), and the Bessel function of order ν_1 and of the first kind.

The polynomial set $\{\bar{R}_n(x, y)\}$ happens to be a generalization of several orthogonal and non-orthogonal polynomials such as Hermite, Laguerre, Legendre, Jacobi, Gegenbauer, Sisterceline, Bateman's and discrete polynomials [see Erdelyi *et al.* (1953) p. 225(9), p. 226(5)].

Fox Example

I. *Hermite polynomials* — For $m = m_4 = \beta = 1 = \mu = x = r_1 = l$;

$$p = 0 = q = l_1 = \alpha_1 = \nu_1 = b_1;\ v = -1;\ \alpha = 1/2;\ y = Y^2,$$

we get from (1.1)

$$\bar{R}^{1;-;-;-1;1/2;1;-;-}_{n;1;-;0;0;(0,1);1;1}(1, Y^2) = \frac{Y^{2n}}{n!}\,{}_2F_0\left[-n, -n+\tfrac{1}{2}; -; -\frac{1}{Y^2}\right]$$

$$= \frac{H_{2n}(Y)}{n!\,2^{2n}}.$$

II. *Laguerre polynomials* — On putting $\alpha = 1 + \lambda$; $m = m_4 = 1 = \beta = v = x$; $P = 0 = q = \alpha_1 = \nu_1 = b_1 = l_2$; $\mu = -1$; $r_1 = 1 = l_1$ in eqn. (1.1), and converting the order of summation (i.e. replacing k by $n - k$), we obtain

$$\bar{R}^{1;-;-;-;1;1+\lambda;-1;-;-}_{n;1;1;0;0;(0,1);1;1}(1, y) = \frac{(1+\lambda)_n}{n!}\,{}_1F_1\,[-n; 1+\lambda; y]$$

$$= L_n^{(\lambda)}(y).$$

III. *Legendre polynomials* — For $\alpha = 1 = v = x = m = m_4 = l_2 = r_1 = l_1 = P$; $q = 0 = b_1 = \alpha_1 = \nu_1$; $y = Y - 1$; $\mu = 2 = \beta$; $a_1 = \frac{1}{2}$; $A_1 = 1$, we get from eqn. (1.1)

$$\bar{R}^{1;-;-;-;1;2;(1/2,1);1}_{n;2;1;0;0;(0,1);1;1}(1, Y-1) = \frac{(1/2)_n\,2^n(Y-1)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n, n; & \\ & \dfrac{2}{1-Y} \\ -2n\ ; & \end{matrix}\right]$$

$$= P_n(Y).$$

Similarly, we may get the results for other polynomial systems.

We obtain the relation for $\{\bar{R}_n(x, y)\}$ in an earlier paper (Singh and Pandey 1977, eqn. (2.1), p. 380),

$$\bar{R}_n(x, y) = \sum_{k=0}^{[n/m_4]}\sum_{s=0}^{[n/2m_3]} \frac{[(M_1(i, j))]_{n-m_4k-2m_3s}\,[1-(M_2(i, j))]_{n-m_4k-2m_3s}}{k!\,(n-m_4k-2m_3s)!\;s!\;[(N_1(i, j))]_{n-m_4k-2m_3s}} \times$$

(*equation continued on p.* 616)

$$\times \frac{\nu^k (\mu E)^{n-m_4k-2m_3s} \left(-\frac{1}{4}\right)^s (\alpha_1)^{2s} \, y^{r_1(n-m_4k-2m_3s)+2m_2s}}{[1-(N_2(i,j))]_{n-m_4k-2m_3s} \, x^{mk-2m_1s} (\nu_1+1)_s}$$

$$\times \frac{(\alpha+\beta b_1)_{n\beta-m_4\beta k-2m_3\beta s+k}}{(\alpha+\beta b_1)_{n\beta-m_4\beta k-2m_3\beta s}}. \qquad \ldots(1.2)$$

For convenience the following notations have been used for brevity :

$$(m) = 1, 2, 3, \ldots, m$$

$$(a_p) = a_1, a_2, \ldots, a_p$$

$$[(a_p)]_n = (a_1)_n \, (a_2)_n \, (a_3)_n \ldots (a_p)_n$$

$$[(M_1(i,j)]_n = \prod_{j=1}^{l_2} \prod_{i=1}^{A_j} \left(\frac{i-a_j+A_jb_1}{A_j}\right)_n$$

$$[1-(M_2(i,j))]_n = \prod_{j=l_2+1}^{p} \prod_{i=1}^{A_j} \left(1-\frac{i-1+a_j-A_jb_1}{A_j}\right)_n$$

$$[(N_1(i,j))]_n = \prod_{j=l_1+1}^{q+1} \prod_{i=1}^{B_j} \left(\frac{i-b_j+B_jb_1}{B_j}\right)_n$$

$$[1-(N_2(i,j))]_n = \prod_{j=2}^{l_1} \prod_{i=1}^{B_j} \left(1-\frac{i-1+b_j-B_jb_1}{B_j}\right)_n$$

$$E = \frac{(-1)^U \prod_{j=1}^{P} A_j^{A_j}}{(-1)^V \prod_{j=2}^{q+1} B_j^{B_j}}$$

$$\text{where} \qquad U = \sum_{j=1}^{l_1} B_j, \qquad V = \sum_{j=l_2+1}^{P} A_j$$

$$\alpha_1 = 0 = \nu_1 = \bar{\bar{R}}_n(x, y).$$

2. Recurrence Relations of the Particularized Polynomial Set $\{\bar{R}_n(x, y)\}$

Relation in y

From (1.2), we arrive at

$$\bar{\bar{R}}_n(x^{\xi}, y^{\eta}) = \sum_{k=0}^{[n/m_4]} \frac{[(M_1(i,j))]_{n-m_4k} \, [1-(M_2(i,j))]_{n-m_4k} \, \nu^k}{k! \, (n-m_4k)! \, [(N_1(i,j))]_{n-m_4k} \, [1-(N_2(i,j))]_{n-m_4k}}$$

$$\times \frac{(\alpha+\beta b_1)_{n\beta+(1-m_4\beta)k} \, (\mu E)^{n-m_4k} \, y^{(n-m_4k)r_1\eta}}{(\alpha+\beta b_1)_{n\beta-m_4\beta k} \, x^{mk\xi}}$$

Differentiating partially with respect to y, we have

$$\frac{\partial}{\partial y}\,\bar{\bar{R}}_n(x^{\xi}, y^{\eta}) = \mu E r_1 \eta\, y^{r_1\eta-1}\,\frac{[(M_1(i,j))]\,[1-(M_2(i,j))]}{[(N_1(i,j))]\,[1-(N_2(i,j))]}$$

$$\times \sum_{k=0}^{[(n-m_4)/m_4]} \frac{[(M_1(i,j))+1]_{n-1-m_4k}\,[2-(M_2(i,j))]_{n-1-m_4k}}{k!\,(n-1-m_4k)!\,[(N_1(i,j))+1]_{n-1-m_4k}}$$

$$\times \frac{v^k(\mu E)^{n-1-m_4k}\, y^{(n-1-m_4k)r_1\eta}(\alpha+\beta b_1+\beta)_{\overline{n-1-m_4k}\beta+k}}{[2-(N_2(i,j))]_{n-1-m_4k}\, x^{mk\xi}(\alpha+\beta b_1+\beta)_{\overline{n-1-m_4k}\beta}}$$

or

$$\left(y^{1-r_1\eta}\frac{\partial}{\partial y}\right)\bar{\bar{R}}_n(x^{\xi}, y^{\eta}) = \mu E r_1\eta\,\frac{[(M_1(i,j))]\,[1-(M_2(i,j))]}{[(N_1(i,j))]\,[1-(N_2(i,j))]}$$

$$\times\, \bar{\bar{R}}^{\alpha+\beta;((a_p,A_p)),(1))}_{n-1;((b_{q+1},B_{q+1})),(1))}(x^{\xi}, y^{\eta}). \qquad \ldots(2.1)$$

Now, differentiating (2.1) successively s_1-times, we have

$$\left(y^{1-r_1\eta}\frac{\partial}{\partial y}\right)^{s_1}[\bar{\bar{R}}_n(x^{\xi}, y^{\eta})] = (\mu E r_1\eta)^{s_1}\,\frac{[(M_1(i,j))]_{s_1}\,[1-(M_2(i,j))]_{s_1}}{[(N_1(i,j))]_{s_1}\,[1-(N_2(i,j))]_{s_1}}$$

$$\times\, \bar{\bar{R}}^{\alpha+\beta s_1;((a_p,A_p)),(1))}_{n-s_1;((b_{q+1},B_{q+1})),(1))}(x^{\xi}, y^{\eta}). \qquad \ldots(2.2)$$

Special cases of (2.2) — (i) On taking $m = m_4 = \beta = 1 = \mu = x = r_1 = l_1$; $P = 0 = q = l_2 = b_1$; $v = -1$; $\alpha = \frac{1}{2}$; $y = Y^2$; $\xi = 1 = \eta$, we obtain the known relation for the Hermite polynomials:

$$\frac{d^{s_1}}{dy^{s_1}}\,H_{2n}(Y) = \frac{2^{2s_1}(n)!}{(n-s_1)!}\,H_{2n-2s_1}(Y).$$

(ii) On making the substitutions $\alpha = 1+\lambda$; $m = m_4 = 1 = \beta = x$;

$$P = 0 = q = b_1 = l_2;\ \mu = -1;\ r_1 = 1 = l_1 = \xi = \eta = v,$$

we get the known result for Laguerre polynomials:

$$\frac{d^{s_1}}{dy^{s_1}}\,L_n^{(\lambda)}(y) = (-1)^{s_1}L_{n-s_1}^{(\lambda+s_1)}(y).$$

(iii) On setting the values $\alpha = 2w$; $v = x = 1 = m = m_4 = l_2 = p = A_1$; $q = 0 = b_1$; $a_1 = 1-w$; $\mu = 2 = \beta$; $y = Y-1$; $r_1 = 1 = l_1 = \xi = \eta$, we arrive at the known results for Gegenbauer polynomials:

$$\frac{d^{s_1}}{dy^{s_1}}\,C_n^{w}(Y) = \frac{(w)_{n+s_1}\,(-n)_{s_1}\,C_n^{w-s_1}(Y)}{(w-s_1)_n\,(1-w-s_1-n)_{s_1}\,(Y-1)^{s_1}}.$$

(iv) On putting $\alpha = 1 + b; b_1 = 0; v = x = m = 1 = m_4 = q;$

$$\mu = 2 = \beta = p = l_2; r_1 = 1 = l_1 = \xi = \eta; y = Y - 1; A_i(i = 1, 2);$$

$$B_2 = 1; b_2 = -a; a_1 = \tfrac{1}{2}(1 - a - b); a_2 = (-a - b),$$

we get the known relation for Jacobi polynomials:

$$\frac{d^{s_1}}{dy^{s_1}} P_n^{(a,b)}(Y) = 2^{-s_1}(1 + a + b + n)_{s_1} P_{n-s_1}^{(a+s_1, b+s_1)}(Y).$$

(v) For $\alpha = 1 + s; m = m_4 = 1 = x = \mu = r_1 = l_1; v = -1;$

$$p = 0 = q = \beta = l_2 = b_1; \xi = 1 = \eta,$$

the eqn. (2.2) reduces to the known result of Srivastava polynomials (Singh 1964):

$$\frac{d^{s_1}}{dy^{s_1}} A_n^{(s)}(y) = A_{n-s_1}^{(s)}(y).$$

Relation in x

Differentiating (1.2) with respect to x, we have

$$\frac{\partial}{\partial x} \bar{\bar{R}}_n(x^{m_4\xi/m}, y^\eta) = \sum_{k=1}^{[n/m_4]} \frac{-m_4\xi\,[(M_1(i,j))]_{n-m_4k}\,[1 - (M_2(i,j))]_{n-m_4k}}{(k-1)!\,(n - m_4k)!\,[(N_1(i,j))]_{n-m_4k}}$$

$$\times \frac{v^k(\mu E)^{n-m_4k}\, y^{r_1\eta(n-m_4k)}(\alpha + \beta b_1)_{n\beta+(1-m_4\beta)k}}{[1 - (N_2(i,j))]_{n-m_4k}\, x^{m_4\xi k-1}(\alpha + \beta b_1)_{n\beta - m_4\beta k}}$$

$$= - m_4\xi v \sum_{k=0}^{[(n-m_4)/m_4]} \frac{[(M_1(i,j))]_{n-m_4-m_4k}\,[1 - (M_2(i,j))]_{n-m_4-m_4k}}{k!\,(n - m_4 - m_4k)!\,[(N_1(i,j))]_{n-m_4-m_4k}}$$

$$\times \frac{(\alpha + \beta b_1 + \overline{n - m_4}\beta + \overline{1 - m_4}\beta k)\, v^k(\mu E)^{n-m_4-m_4k}\, y^{r_1\eta(n-m_4-m_4k)}}{[1 - (N_2(i,j))]_{n-m_4-m_4k}\, x^{(m_4k+m)\xi+1}}$$

$$\times \frac{(\alpha + \beta b_1)_{\overline{n-m_4}\beta + \overline{1-m_4}\beta k}}{(\alpha + \beta b_1)_{n\beta - m_4\beta k - m_4\beta}}.$$

The above may be written as

$$\frac{\partial}{\partial x} \bar{\bar{R}}_n(x^{m_4\xi/m}, y^\eta) + m_4\xi v\,[\alpha + \beta b_1 + \overline{n - m_4}\beta]\, x^{-m_4\xi-1}$$

$$\times \bar{\bar{R}}_{n-m_4}(x^{m_4\xi/m}, y^\eta) + v(m_4\beta - 1)\, x^{-m_4\xi} \frac{\partial}{\partial x} \bar{\bar{R}}_{n-m_4}(x^{m_4\xi/m}, y^\eta) = 0. \quad \ldots(2.3)$$

Special case — On taking $\alpha = a + n$; $\beta = 1 = \mu = y = m = m_4 = r_1 = l_1$; $v = 1 = \xi = \eta$; $l_2 = p$; $(a_p) = 1 - (\alpha_p)$, $A_1 = 1 = A_2 = A_3 = \ldots = A_p$, $(b_{q+1}) = 1 - (\beta_q)$,

$$B_2 = 1 = B_3 = \ldots = B_{q+1};\ b_1 = 0$$

in (2.3), we obtain the known result of the Shively Toscano polynomials:

$$D\left[\frac{1}{x^n} S_n(z)\right] = -\frac{(a+2n-1)}{x^{n+1}} S_{n-1}(x)$$

where

$$D \equiv \frac{d}{dy}.$$

Pure Recurrence Relation

On differentiating (1.2), partially with respect to y and then breaking $n - m_4k$ in two parts, we obtain

$$\frac{\partial}{\partial y}\bar{\bar{R}}_n(x^\xi, y^\eta) = \frac{r_1 n\eta}{y}\sum_{k=0}^{[n/m_4]} \frac{[(M_1(i,j))]_{n-m_4k}\,[1-(M_2(i,j))]_{n-m_4k}}{k!\,(n-m_4k)!\,[(N_1(i,j))]_{n-m_4k}}$$

$$\times \frac{v^k(\mu E)^{n-m_4k}\, y^{r_1(n-m_4k)\eta}(\alpha+\beta b_1)_{n\beta+(1-m_4\beta)k}}{[1-(N_2(i,j))]_{n-m_4k}\, x^{mk\xi}(\alpha+\beta b_1)_{n\beta-m_4\beta k}}$$

$$= \frac{r_1 n\eta}{y}\bar{\bar{R}}_n(x^\xi, y^\eta) - r_1\eta m_4 v\,\frac{[\alpha+\beta b_1+\overline{n-m_4}\beta(1-m_4\beta)]}{y}$$

$$\times \sum_{k=0}^{[(n-m_4)/m_4]} \frac{[(M_1(i,j))]_{n-m_4-m_4k}\,[1-(M_2(i,j))]_{n-m_4-m_4k}}{k!\,(n-m_4-m_4k)!\,[(N_1(i,j))]_{n-m_4-m_4k}}$$

$$\times \frac{v^k(\mu E)^{n-m_4-m_4k}\, y^{r_1(n-m_4-m_4k)\eta}(\alpha+\beta b_1)_{\overline{n-m_4}\beta+\overline{1-m_4}\beta k}}{[1-(N_2(i,j))]_{n-m_4-m_4k}\, x^{mk\xi+m\xi}(\alpha+\beta b_1)_{n\beta-m_4\beta k-m_4\beta}}.$$

The above can be written as

$$\frac{\partial}{\partial y}\bar{\bar{R}}_n(x^\xi, y^\eta) = \frac{r_1 n\eta}{y}\bar{\bar{R}}_n(x^\xi, y^\eta) - m_4 v r_1\eta x^{-m\xi}\frac{[\alpha+\beta b_1+\overline{n-m_4}\beta]}{y}$$

$$\times \bar{\bar{R}}_{n-m_4}(x^\xi, y^\eta) + v(1-m_4\beta)\, x^{-m\xi}\frac{\partial}{\partial y}\bar{\bar{R}}_{n-m_4}(x^\xi, y^\eta)$$

$$+ \frac{(m_4\beta-1)(n-m_4)\, r_1\eta v x^{-m\xi}}{y}\bar{\bar{R}}_{n-m_4}(x^\xi, y^\eta). \qquad \ldots(2.4)$$

From recurrence relations (2.1) and (4.1), we can easily arrive at the following result:

$$\mu E r_1 \eta y^{r_1 \eta - 1} \frac{[(M_1(i,j))]\,[(1-(M_2(i,j))]}{[(N_1(i,j))]\,[1-(N_2(i,j))]} \bar{\bar{R}}^{\alpha+\beta;((a_p,A_p)),(1))}_{n-1;((b_{q+1},B_{q+1})),(1))}(x^{\xi}, y^{\eta})$$

$$= \frac{r_1 n\eta}{y} \bar{\bar{R}}_n(x^{\xi}, y^{\eta}) - m_4 r_1 \eta v x^{-m\xi} \frac{[\alpha+\beta b_1 + \overline{n - m_4 \beta}]}{y} \bar{\bar{R}}_{n-m_4}(x^{\xi}, y^{\eta})$$

$$+ v x^{-m\xi}(1 - m_4\beta)\, \mu E r_1 n\eta \frac{[(M_1(i,j))]\,[1-(M_2(i,j))]}{[(N_1(i,j))]\,[1-(N_2(i,j))]}$$

$$\times \bar{\bar{R}}^{\alpha+\beta;((a_p,A_p)),(1))}_{n-m_4-1;((b_{q+1},B_{q+1})),(1))}(x^{\xi}, y^{\eta}) - \frac{(n-m_4)\, r_1 \eta v (1-m_4\beta)\, x^{-m\xi}}{y}$$

$$\times \bar{\bar{R}}_{n-m_4}(x^{\xi}, y^{\eta}). \quad \ldots(2.5)$$

Special cases of (2.5) — On using the values of the parameters given in (2.1, I, II) we arrive at the following well-known results for Hermite and Laguerre polynomials (Rainville 1960, pp. 188, 203):

(i) $H_{2n}(Y) = 2YH_{2n-1}(Y) - 2\,(2n-1)\,H_{2n-2}(Y)$

(ii) $yL^{(\lambda+1)}_{n-1}(y) = (\lambda + n)\,L^{(\lambda)}_{n-1}(y) - nL^{(\lambda)}_{n}(y)$.

Acknowledgement

The author expresses his gratitude to the referee for his valuable comments.

References

Erdelyi, A., *et al.* (1953). Higher Transcendental Functions, Vol. II. Bateman Manuscript Project, California Institute of Technology.

Fox, C. (1961). The G- and H-functions as symmetrical Fourier kernels. *Trans. Am. math. Soc.*, **98**, 395–425.

Rainville, E. D. (1960). Special Functions. The Macmillan Company, New York.

Singh, R. P. (1964). Some polynomials of Srivastava related to the Laguerre polynomials. *Math. Japon*, **9**, 5–9.

Singh, R. B., and Pandey, R. N. (1977). A different representation of the generalized polynomials $\bar{R}_n(x, y)$. *Acta-Ciencia Indica*, **3**, 378–82.