Indian J. pure appl. Math., 12(5) : 621–627, May 1981

FUNCTIONS OF BOUNDED RADIUS ROTATION

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(*Received* 13 *March* 1980)

If $f(z)$ is a function of radius rotation at most $k\pi (k \geqslant 2)$, then it is shown that

$$F(z) = \frac{2}{z} \int_0^z f(t)\, dt$$

is starlike for $|z| < (k - \sqrt{k^2 - 4})/2$. Conversely an estimate is obtained for the radius of starlikeness of the class of functions

$$f(z) = 2^{-1}(zF(z))'$$

where $F(z)$ is a function of radius rotation atmost $k\pi$, $k \geqslant 2$. This estimate is sharp for $k \geqslant 4$.

§1. All functions $f(z)$ are analytic in the open unit disc D and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$ unless stated otherwise.

P will denote the class of functions $P(z)$ which are analytic, have positive real part in D, and satisfy $P(0) = 1$

M_k will denote the class of real valued functions $m(t)$ of bounded variation on $[-\pi, \pi]$ which satisfy the conditions

$$\int_{-\pi}^{\pi} dm(t) = 2, \quad \int_{-\pi}^{\pi} |\, dm(t)\,| \leqslant k. \qquad \text{...(1)}$$

M_2 is clearly the class of non decreasing functions on $[-\pi, \pi]$ satisfying

$$\int_{-\pi}^{\pi} dm(t) = 2.$$

If $m(t) \in M_k$ with $k > 2$ we can write $m(t) = \alpha(t) - \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are both non decreasing functions on $[-\pi, \pi]$ and satisfy

$$\int_{-\pi}^{\pi} d\alpha(t) \leqslant [k/2] + 1 \quad \text{and} \quad \int_{-\pi}^{\pi} d\beta(t) \leqslant [k/2] - 1. \qquad \text{...(2)}$$

Let S^* denote the class of starlike, univalent functions namely those functions $f(z)$ which map D on to a domain that is starlike with respect to the origin.

*Research supported by UGC Grant F. 25-3(8772)/77 SRI.

It is known that $f(z) \in S^*$ if and only if

$$f(z) = z \exp \int_{-\pi}^{\pi} - \log (1 - ze^{-it})\, dm(t) \qquad \text{...(3)}$$

for some $m(t) \in M_2$.

As in Pinchuk (1971) we can generalise the class S^* by allowing $m(t)$ to range over the class M_k. More precisely a function $f(z)$ is said to be in the class U_k if

$$f(z) = z \exp \int_{-\pi}^{\pi} - \log (1 - ze^{-it})\, dm(t) \qquad \text{...(4)}$$

for some $m(t) \in M_k$.

We also note that U_k consists of functions $f(z)$ which satisfy

$$\int_{-\pi}^{\pi} \left| \operatorname{Re} re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leqslant k\pi \quad \text{for} \quad r < 1. \qquad \text{...(5)}$$

Geometrically the condition is that the total variation of the angle which the radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded above by $k\pi$ as z describes the circle $|z| = r$ for each $r < 1$. Thus U_k is the class of functions with radius rotation bounded by $k\pi$. It may be noted that $U_2 = S^*$

Definition — P_k denotes the class of functions which are analytic in D and have the representation

$$P(z) = \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, dm(t) \qquad \text{...(6)}$$

where $m(t) \in M_k$.

Clearly we have $P_2 = P$. $f \in U_k$ if and only if $\dfrac{zf'}{f} \in P_k$.

Libera (1965) showed that if $f(z) \in U_2$ then

$$F(z) = \frac{2}{z} \int_0^z f(t)\, dt \qquad \text{...(7)}$$

also belongs to U_2. In the converse direction, Livingston (1966) has studied the mapping properties of the function.

$$f(z) = 2^{-1} [zF(z))]' \qquad \text{...(8)}$$

where $F(z) \in U_2$. The object of this paper is to generalise these results of Libera and Livingston by choosing instead of U_2 the class U_k of functions of bounded radius rotation. More precisely we ask for the largest disc in which every function of the form (7) or (8) for f or $F \in U_k$ is starlike. Extensions of the results of Libera and Livingston in other directions were made by Bernardi (1970), Nikolaéva and Repnina (1972), Karunakaran (1975) and Karunakaran and Ziegler (1980).

§2. *Theorem* 1 — Let $f(z) \in U_k$ and $F(z) = \frac{2}{z} \int_0^z f(t)\, dt$. Then $F(z)$ is starlike for $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$.

PROOF: Examining the proof of Lemma 1 in Libera (1965) we easily find that if N and D are regular in the unit disc, D maps $|z| < r$ onto a many sheeted region which is starlike with respect to origin, and if $\mathrm{Re}\,(N'/D') > 0$ for $|z| < r$ then $\mathrm{Re}\,(N/D) > 0$ for $|z| < r$. Now $F(z) = 2\sigma(z)/z$ where $\sigma(z) = \int_0^z f(t)\, dt$. We note that $\sigma(z)$ is two valently starlike with respect to the origin in $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$ because of the fact that $f \in U_k$ is starlike (univalently) in $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$ (see Pinchuk 1971) and the proof of Lemma 2 of Libera (1965). So

$$\frac{zF'(z)}{F(z)} = [z\sigma'(z)/\sigma(z)] - 1$$

$$= N(z)/D(z) \quad \text{where} \quad N(z) = z\sigma'(z) - \sigma(z)$$

$D(z) = \sigma(z)$. Since $(N'/D') = z\sigma''(z)/\sigma'(z) = zf'/f$ we have $\mathrm{Re}\,(N'/D') > 0$ for $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$. Thus $\mathrm{Re}\,(N/D) > 0$ for $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$ and hence $F(z)$ is starlike for $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$.

We now proceed to examine the behaviour of $f(z)$ given by (8) when $F(z) \in U_k$.

Theorem 2 — Let $F(z) \in U_k$. Define $f(z) = \frac{1}{2}[zF(z)]'$, then $F(z)$ is starlike $|z| < R$ where R is given by the smallest positive root of the equation

$$4 - 8kr + r^2(8 - 2A_k + 3k^2) - 4kr^3 = 0$$

where A_k is $2(k - 1)$ or $(k^2 - 4)/2$ according as $k \leqslant 4$ or $k \geqslant 4$. The radius is best possible if $k \geqslant 4$.

We need the following lemma.

Lemma — Let $q(z)$ be analytic in the unit disc, $q(0) = 1$ and satisfy

$$\int_0^{2\pi} \left| \mathrm{Re}\, \frac{q(re^{i\theta}) - \rho}{1 - \rho} \right| d\theta \leqslant k\pi$$

for $r < 1$ and $0 \leqslant \rho \leqslant \frac{1}{2}$. Let the following differential equation

$$f'(z) + zf''(z) = q(z)\, f'(z)$$

admits a solution $f(z)$ with the initial condition $f(0) = 0$ and $f'(0) = 1$. Then for $C \geqslant 0$ and $|z| = r < r_0$ we have

$$\operatorname{Re}\left\{Cq(z) + \frac{zq'(z)}{q(z)}\right\} \geqslant L(a - d).$$

where $$L(R) = \frac{-\rho}{1-\rho} + \left[C + \frac{1}{2(1-\rho)}\right] R + \frac{2\rho - 1}{2(1-\rho)} \frac{1}{R} - \frac{A_k(1-\rho)\, r^2}{(1-r^2)^2\, R}$$

and $$a = 1 + \frac{(2-2\rho)\, r^2}{1-r^2},\quad d = \frac{kr(1-\rho)}{1-r^2},$$ r_0 is the least positive root of

$$1 - k(1-\rho)\, r + (1-2\rho)\, r^2 = 0,$$

$$A_k = 2(k-1) \quad \text{if} \quad 2 \leqslant k \leqslant 4 \quad \text{and} \quad A_k = \frac{k^2-4}{2} \quad \text{if} \quad k \geqslant 4.$$

PROOF : Let $f_0(z)$ be defined by

$$f'(z) = [f_0'(z)]^{1-\rho}$$

where $f(z)$ is the solution of differential equation $f'(z) + zf''(z) = q(z)\, f'(z)$. Then $f_0(z)$ is a function of bounded boundary rotation. Hence by Robertson (1969) we get

$$F_0(z) = \frac{f_0\left[\dfrac{z+a}{1+\bar{a}z}\right] - f_0(a)}{f_0'(a)\,(1 - |a|^2)},\quad |a| < 1$$

is also a function of bounded boundary rotation.

We also know (Robertson 1969) that $|F_0''(0)| \leqslant k$. Hence by a simple computation we have

$$\left|\frac{f''(a)}{f'(a)} - \frac{2(1-\rho)\,\bar{a}}{1-|a|^2}\right| \leqslant \frac{k(1-\rho)}{1-|a|^2}.$$

Since a is arbitrary in the unit disc we have

$$\left|\frac{f''(z)}{f'(z)} - \frac{2(1-\rho)\,\bar{z}}{1-|z|^2}\right| \leqslant \frac{k(1-\rho)}{1-|z|^2}.$$

Substituting

$$\frac{zf''}{f'} = q(z) - 1 \quad \text{we get,}$$

$|q(z) - a| \leqslant d$ where a and d are as stated in the Lemma.

We also know (Moulis 1972) that the Schwarzian derivative $\{f_0, z\}$ of f_0 satisfies

$$|\{f_0, z\}| \leqslant \frac{A_k}{(1-|z|^2)^2}$$

where
$$A_k = \begin{cases} (k^2-4)/2, & k \geqslant 4 \\ 2(k-1) & 2 \leqslant k \leqslant 4. \end{cases}$$

Thus
$$\left|\left[\frac{f_0''(z)}{f_0'(z)}\right]' - \tfrac{1}{2}\left[\left[\frac{f_0''(z)}{f_0'(z)}\right]^2\right]\right| \leqslant \frac{A_k}{(1-r^2)^2}.$$

But

$$\frac{f''(z)}{f'(z)} = (1-\rho)\, f_0''(z)/f_0'(z).$$

Hence
$$\left|\frac{1}{(1-\rho)}\left[\frac{f''(z)}{f'(z)}\right]' - \tfrac{1}{2}\,\frac{1}{(1-\rho)^2}\left(\frac{f''(z)}{f'(z)}\right)^2\right| \leqslant \frac{A_k}{(1-r^2)^2}.$$

Again using $zf''(z)/f'(z) = q(z) - 1$ and simplifying we get

$$\left|\frac{zq'(z)}{q(z)} + \frac{\rho}{1-\rho} + \frac{(1-2\rho)}{2-2\rho}\,\frac{1}{q(z)} - \tfrac{1}{2}\,\frac{q(z)}{1-\rho}\right| \leqslant \frac{A_k(1-\rho)\,r^2}{(1-r^2)^2\,|q(z)|}$$

Hence
$$\operatorname{Re}\left\{Cq(z) + \frac{zq'(z)}{q(z)}\right\} \geqslant \frac{-\rho}{1-\rho} + \left[C + \frac{1}{2(1-\rho)}\right]\operatorname{Re} q(z)$$
$$+ \frac{(2\rho-1)}{2-2\rho}\operatorname{Re}\frac{1}{q(z)} - \frac{A_k(1-\rho)\,r^2}{(1-r^2)^2\,|q(z)|}$$

Now we can write $q(z) = a + u + iv$ so that $u^2 + v^2 < d^2$

Let also $R^2 = (a+u)^2 + v^2$. Substituting for Re q, Re $\dfrac{1}{q}$ and $|q|$ we get

$$\operatorname{Re}\left\{Cq(z) + z\,\frac{q'(z)}{q(z)}\right\} \geqslant S(u, v)$$

where
$$S(u, v) = \frac{-\rho}{1-\rho} + \left[C + \frac{1}{2(1-\rho)}\right](a+u)$$
$$+ \frac{(2\rho-1)}{2-2\rho}\left\{\frac{a+u}{R^2}\right\} - \frac{A_k(1-\rho)\,r^2}{(1-r^2)^2\,R}.$$

Now
$$\frac{\partial S}{\partial v}(u, v) = vR^{-4}\left\{2(a+u)\,\frac{(1-2\rho)}{2-2\rho} + \frac{A_k(1-\rho)\,r^2}{(1-r^2)^2}\,R\right\}.$$

We note that $a + u > 0$ if $a - d > 0$ or $r < r_0$ where r_0 is the least positive root of the equation

$$1 - k(1-\rho)\,r + (1-2\rho)\,r^2 = 0.$$

Hence $S(u, v)$ is an increasing function of v and so its minimum is when $v = 0$. The minimum value is given by

$$S(u, 0) = S(R - a, 0) = L(R)$$

where $L(R)$ is as stated in the lemma. Further $L(R)$ as a function of R increases and so its minimum value is $L(a - d)$ since $R > (a - d)$ holds.

Proof of the theorem :

We have

$$f(z) = \tfrac{1}{2}[zF'(z) + F(z)].$$

Hence

$$\frac{zf'z}{f(z)} = \frac{zF'(z)}{F(z)} + \frac{zq'(z)}{q(z)}.$$

where $$q(z) = \tfrac{1}{2}\left[\frac{zF'(z)}{F(z)} + 1\right].$$

Since $F(z) \in U_k$ $\dfrac{zF'(z)}{F(z)} \in P_k$. Thus $q(z) = \tfrac{1}{2}\left[\dfrac{zF'(z)}{F(z)} + 1\right]$ satisfies the conditions of the lemma with $\rho = \tfrac{1}{2}$. Also the differential equation $f'(z) + zf''(z) = q(z) f'(z)$ admits the solution given by $f(z) = \int_0^z [F(t)/t]^{1/2}\, dt$, where we choose the principal branch for the multivalued function.

$$\frac{zf'(z)}{f(z)} = 2q(z) + \frac{zq'(z)}{q(z)} - 1$$

Applying lemma with $C = 2$ and $\rho = \tfrac{1}{2}$ we see that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geqslant 0 \quad \text{if} \quad L(a - d) - 1 > 0$$

where $$a = \frac{1}{1 - r^2},\ d = \frac{kr}{2(1 - r^2)} \quad \text{and} \quad r < r_0 \quad \text{with} \quad r_0 = 2/k.$$

A simple calculation shows that this will be true for r less than the least positive root of the equation given in the theorem depending on k. For $k > 4$ function defined by

$$F(z) = \frac{z(1 + z)^{k/2-1}}{(1 - z)^{k/2+1}}$$

shows that $\dfrac{zf'(z)}{f(z)} = 0$ for $z = -R$ where $f(z) = [zF(z)]'/2$.

This completes the proof of the theorem.

Using the fact that $f(z) \in U_k$ for $|z| < R$ if and only if $\int_0^z f(t)/t\, dt$ is function of boundary rotation atmost $k\pi$ we can easily get the following two theorems.

Theorem 3 — Let $f(z)$ be a function of boundary rotation atmost $k\pi$. Then $F(z) = [2/z] \int_0^z f(t)\, dt$ is convex in the disc $|z| < \frac{1}{2}(k - \sqrt{k^2 - 4})$.

Theorem 4 — Let $F(z)$ be a function of boundary rotation atmost $k\pi$. Then $f(z) = \frac{1}{2}(zF)'$ is convex in the disc $|z| < R$ where R is the least possible root of the equation

$$4 - 8kr + r^2(8 - 2A_k + 3k^2) - 4kr^3 = 0$$

where $A_k = 2(k - 1)$ or $(k^2 - 4)/2$ according as $k \leqslant 4$ or $k \geqslant 4$

The radius is best possible if $k \geqslant 4$.

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