Indian J. pure appl. Math., **12**(5) : 628-633, May 1981

ON A CLASS OF SPIRAL-LIKE FUNCTIONS

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(*Received* 9 *June* 1980)

Let μ, ν, α and $\beta, |\beta| < \frac{\pi}{2}$ be fixed real numbers. We denote by $S(\alpha, \beta, \mu, \nu)$ the class of normalized functions regular in the unit disc E with $f(z)/z$, $f'(z)$, $1 + \frac{zf''(z)}{f'(z)}$ not equal to zero in E, which satisfy the condition

$$\operatorname{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right)^{\mu}\left((e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)^{\nu} > 0,$$

$z \in E$ and the powers appearing are meant as principal values. In this paper we prove that every member of $S(\alpha, \beta, \mu, \nu)$ is a β-spiral function for $(\mu, \nu) \in G$ where

$$G = [(\mu, \nu) : 4n + 1 \leqslant \mu + \nu \leqslant 4n + 3, n \in I]$$
$$\cup [(\mu, 0) : |\mu| \geqslant 1] \cup [(0, \nu) : |\nu| \geqslant 1]$$

and where I is the set of integers. Some known results of different authors follow as special cases from our results.

§1. Let A denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular in the unit disc $E = \{z : |z| < 1\}$. By S, S^*, K and S_α we denote the subclasses of A which are respectively, univalent, starlike (with respect to the origin), convex and α-convex in E. For a given real number β, $|\beta| < \pi/2$, we denote by $S_p(\beta)$ the subclass of A whose members satisfy the condition

$$\operatorname{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in E. \qquad \ldots(1)$$

It is well known that every member of $S_p(\beta)$ belongs to the class S. The members of this class are called β-spiral functions. A function $f \in A$ is said to be β-spiral convex if and only if it satisfies the condition

$$\operatorname{Re}\left\{\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right) + i\sin\beta\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in E \qquad \ldots(2)$$

for some fixed β. We denote this class by $S_c(\beta)$. Yoshikawa (1971) introduced this class and proved that $S_c(\beta) \subset S_p(\beta)$. Recently Umarani (1979) has introduced a

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subclass $S_c(\alpha, \beta)$ of $S_p(\beta)$, which for different values of α and β leads to the class of α-convex functions and β-spiral convex functions. A function $f \in A$ with $\frac{f(z) \cdot f'(z)}{z} \neq 0$ in E is said to belong to the class $S_c(\alpha, \beta)$ if and only if it satisfies the inequality.

$$\operatorname{Re} J(\alpha, \beta, f(z)) > 0, \quad z \in E \qquad \ldots(3)$$

where α is any real number and β is fixed, $|\beta| < \pi/2$ and

$$J(\alpha, \beta, f(z)) = (e^{i\beta} - \alpha \cos \beta) \frac{zf'(z)}{f(z)} + \alpha \cos \beta \left(1 + \frac{zf''(z)}{f'(z)}\right). \qquad \ldots(4)$$

Obviously $S_c(\alpha, 0)$ is the class of α-convex fuctions, $S_c(1, \beta)$ is the class of β-spiral convex functions and $S_c(0, \beta)$ is the class of β-spiral functions. He proved that (Umarani 1979).

$$S_c(\alpha, \beta) \subset S_p(\beta) \qquad \ldots(5)$$

Let μ, ν, α and β, $|\beta| < \pi/2$ be fixed real numbers. We denote by $S(\alpha, \beta, \mu, \nu)$ the class of functions for which $f(z)/z$, $f'(z)$, $1 + \frac{zf''(z)}{f'(z)}$ are not zero in E and which satisfy the condition

$$\operatorname{Re}\left(e^{i\beta} \frac{zf'(z)}{f(z)}\right)^{\mu} \left((e^{i\beta} - \alpha \cos \beta) \frac{zf'(z)}{f(z)} + \alpha \cos \beta \left(1 + \frac{zf''(z)}{f'(z)}\right)\right)^{\nu} > 0 \qquad \ldots(6)$$

$z \in E$ and the powers appearing in (6) are meant as principal values. Clearly the class $S(1, 0, \mu, \nu)$ coincides with the class $S(\mu, \nu)$ defined by Miller (1976). Miller proved that $S(\mu, \nu) \subset S^*$, if $(\mu, \nu) \in G$ where

$$G = [(\mu, \nu) : 4n + 1 \leqslant \mu + \nu \leqslant 4n + 3, n \in I] \cup [(\mu, 0) ; |\mu| \geqslant 1] \cup [(0, \nu) : |\nu| \geqslant 1]$$

where I is the set of integers.

In this paper we shall prove that $S(\alpha, \beta, \mu, \nu) \subset S_p(\beta)$ for $(\mu, \nu) \in G$ and $\alpha > 0$. Results of Miller (1976), Lewandowski *et al.* (1974), Yoshikawa (1971) and Singh and Singh (1980) follow as special cases from Theorem 1, proved below. We also find the largest radius $R = R(\alpha, \beta)$ such that every function in $S_p(\beta)$ is in the class $S_c(\alpha, \beta)$ for $|z| < R(\alpha, \beta)$. Results of Ruscheweyh and Singh (1976) and Miller *et al.* (1974) are simple consequences of Theorem 3 given below.

§2. We use the technique due to Miller (1976).

Theorem 1 — $S(\alpha, \beta, \mu, \nu) \subset S_p(\beta)$ if $(\mu, \nu) \in G$ and $\alpha > 0$.

PROOF: For $|\mu| \geqslant 1$, $\nu = 0$ and $|\nu| \geqslant 1$, $\mu = 0$, the cases are trivial. So we are to prove the result for the region

$$4n + 1 \leqslant \mu + \nu \leqslant 4n + 3, \; n \in I, \quad \text{the set of integers.}$$

Let $f \in S(\alpha, \beta, \mu, \nu)$ and suppose

$$e^{i\beta} \frac{zf'(z)}{f(z)} = i \sin \beta + \cos \beta \frac{1 + w(z)}{1 - w(z)}$$

$$= \frac{e^{i\beta} + e^{-i\beta} w(z)}{1 - w(z)}. \qquad \ldots(7)$$

Then clearly $w(z)$ is regular in E and $w(0) = 0$, $w(z) \neq 1$ in E. It is sufficient to show that $| w(z) | < 1$, $(\mu, \nu) \in G$. Differentiating (7) logarithmically we obtain after simple computation

$$\left[e^{i\beta} \frac{zf'(z)}{f(z)}\right]^{\mu} \left[(e^{i\beta} - \alpha \cos \beta) \frac{zf'(z)}{f(z)} + \alpha \cos \beta \left(1 + \frac{zf''(z)}{f(z)}\right)\right]^{\nu}$$

$$= \left[\frac{e^{i\beta} + e^{-i\beta} w(z)}{1 - w(z)}\right]^{\mu} \left[\frac{e^{i\beta} + e^{-i\beta} w(z)}{1 - w(z)} + \frac{2\alpha \cos^2 \beta z w'(z)}{(e^{i\beta} + e^{-i\beta} w(z))(1 - w(z))}\right]^{\nu}$$

$$= \left[\frac{e^{i\beta} + e^{-i\beta} w(z)}{1 - w(z)}\right]^{\mu} \left[\frac{e^{i\beta} + e^{-i\beta} w(z)}{1 - w(z)} + \alpha \cos \beta \frac{zw'(z)}{w(z)} \left(\frac{e^{-i\beta} w(z)}{e^{i\beta} + e^{-i\beta} w(z)} + \frac{w(z)}{1 - w(z)}\right)\right]^{\nu} \qquad \ldots(8)$$

Now we claim that $| w(z) | < 1$, for otherwise by Jack's Lemma (1971) there exists a $z_0 \in E$ such that

$$\max_{|z| \leqslant |z_0|} | w(z_0) | = | w(z_0) | = 1, \; w(z_0) \neq 1, \text{ and}$$

$$\frac{z_0 w'(z_0)}{w(z_0)} = m, \; m \geqslant 1. \qquad \ldots(9)$$

Since $| w(z_0) | = 1$ and $w(z_0) \neq \pm 1$, we have

$$\frac{e^{i\beta} + e^{-i\beta} w(z_0)}{1 - w(z_0)} = Ai, \qquad \ldots(10)$$

where A is a real number and $A \neq 0$.

From (8), (9) and (10), we have

$$\text{Re}\left[e^{i\beta} \frac{z_0 f'(z_0)}{f(z_0)}\right]^{\mu} \left[(e^{i\beta} - \alpha \cos \beta) \frac{z_0 f'(z_0)}{f(z_0)} + \alpha \cos \beta \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)\right]^{\nu} =$$

(equation continued on p. 631)

$$= \operatorname{Re}\left[\frac{e^{i\beta} + e^{-i\beta}w(z_0)}{1 - w(z_0)}\right]^{\mu}\left[\frac{e^{i\beta} + e^{-i\beta}w(z_0)}{1 - w(z_0)} + \frac{z\alpha \cos^2 \beta\, mw(z_0)}{(e^{i\beta} + e^{-i\beta}w(z_0))\,(1 - w(z_0))}\right]^{\nu}$$

$$= \operatorname{Re}\,[Ai]^{\mu}\left[Ai + \frac{i\alpha m}{2A}(1 + A^2 - 2A \sin \beta)\right]^{\nu}$$

$$= \operatorname{Re}\,[Ai]^{\mu}\,[Bi]^{\nu} \quad \text{where} \quad B = \left[A + \frac{\alpha m}{2A}(1 + A^2 - 2A \sin \beta)\right]$$

$$= |\,A\,|^{\mu}\,|\,B\,|^{\nu} \cos\,(\mu + \nu)\,\pi/2$$

$\leqslant 0$ [since $AB > 0$ and $\mu + \nu$ is restricted by the condition $4n + 1 \leqslant \mu + \nu \leqslant 4n + 3$.]

which is a contradiction to our hypothesis. Hence $|\,w(z)\,| < 1$ and by (7) $f \in S_p(\beta)$.

Particular Cases

1. By putting $\beta = 0$, $\alpha = 1$, we obtain the result of Miller (1976) i.e. $S(\mu, \nu) \subset S^*$ for $z \in G$.

2. By putting $\beta = 0$, $\mu = 0$ and $\nu = 1$, we see that all α-convex functions are univalent and starlike. This result was proved by Miller *et al.* (1973).

3. By putting $\mu = 1 - \nu$ and $\alpha = 1$, we obtain the result of Singh and Singh (1980).

4. Putting $\mu = 0$, $\nu = 1$ and $\alpha = 1$, we obtain the relation $S_c(\beta) \subset S_p(\beta)$ which was obtained by Yoshikawa (1971).

In the above theorem we have proved the class relationship $S(\alpha, \beta, \mu, \nu) \subset S(\alpha, \beta, 1, 0)$. We generalize this result in the following theorem :

Theorem 2 — $S(\alpha, \beta, \mu, \nu) \subset S(\alpha, \beta, (\mu - 1)\,t + 1, \nu t)$, $0 \leqslant t \leqslant 1$ and $(\mu, \nu) \in G$.

PROOF : Let $f \in S(\alpha, \beta, \mu, \nu)$ and

$$\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right)^{\mu}\left((e^{i\beta} - \alpha \cos \beta)\frac{zf'(z)}{f(z)} + \alpha \cos \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)^{\nu} = P_1(z) \qquad \ldots(11)$$

Then $\operatorname{Re} P_1(z) > 0$, $z \in E$ and $(\mu, \nu) \in G$.

From Theorem 1, we know that

$$e^{i\beta}\frac{zf'(z)}{f(z)} = P_2(z) \qquad \ldots(12)$$

where $\operatorname{Re} P_2(z) > 0$, $z \in E$, $(\mu, \nu) \in G$.

Taking the tth power of (11) and $(1-t)$th power of (12) and then multiplying we get

$$\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right)^{(\mu-1)t+1}\left((e^{i\beta}-\alpha\cos\beta)\frac{zf'(z)}{f(z)}+\alpha\cos\beta\left(1+\frac{zf''(z)}{f(z)}\right)\right)^{vt}$$

$$=(P_1(z))^t\,(P_2(z))^{1+t}=P_3(z) \quad \text{say}$$

Since $0\leqslant t\leqslant 1$, we have

$$|\arg P_3(z)|\leqslant t\,|\arg P_1(z)|+(1-t)\,|\arg P_2(z)|\leqslant \pi/2,$$

which shows that $\operatorname{Re} P_3(z)>0$. This completes the proof of the theorem.

§3. In this section we shall prove the converse of relation (5). Let $\mathcal{P}$ be the class of functions $p(z)$ regular in the unit disc E with $p(0)=1$, $\operatorname{Re} p(z)>0$ in E. We need the following lemma due to Ruscheweyh and Singh (1976).

Lemma 1 — For $p\in\mathcal{P}$ and $s>0$, we have

$$\operatorname{Re}\left(p(z)+s\frac{zp'(z)}{p(z)+\mu}\right)>0 \text{ for}$$

$$|z|<\frac{|\mu+1|}{[A+(A^2-|\mu^2-1|^2)^{1/2}]^{1/2}},\quad A=2(s+1)^2+|\mu|^2-1.$$

The bound is best possible.

Theorem 3 — Let $\alpha>0$. If $f\in S_p(\beta)$ then $f\in S_c(\alpha,\beta)$ for

$$|z|<\frac{\sec\beta}{[B+(B^2-\sec^4\beta)^{1/2}]^{1/2}},\quad B=2(\alpha+1)^2+\tan^2\beta-1.$$

The result is sharp.

PROOF : Let $f\in S_p(\beta)$, then

$$e^{i\beta}\frac{zf'(z)}{f(z)}=\cos\beta p(z)+i\sin\beta \qquad \ldots(13)$$

where $p(z)\in\mathcal{P}$.

Taking logarithmic differentiation of (13), we get

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}+\cos\beta\frac{zp'(z)}{\cos\beta p(z)+i\sin\beta} \qquad \ldots(14)$$

Using (13) and (14) we conclude that

$$J(\alpha,\beta,f(z))\equiv(e^{i\beta}-\alpha\cos\beta)\frac{zf'(z)}{f(z)}+\alpha\cos\beta\left(1+\frac{zf''(z)}{zf'(z)}\right)$$

$$=\cos\beta p(z)+i\sin\beta+\alpha\cos^2\beta\frac{zp'(z)}{\cos\beta p(z)+i\sin\beta} \qquad \ldots(15)$$

Then

$$\operatorname{Re} J(\alpha, \beta, f(z)) = \cos \beta \operatorname{Re}\left[p(z) + \alpha \frac{zp'(z)}{p(z) + i \tan \beta} \right]. \qquad \ldots(16)$$

By Lemma 1, $\operatorname{Re} J(\alpha, \beta, f(z)) > 0$ for

$$| z < | \frac{\sec \beta}{[B + (B^2 - \sec^4 \beta)^{1/2}]^{1/2}}.$$

The result is sharp for the function $f(z) = z(1 - z)^{-2 \exp(-i\beta) \cos \alpha}$.

Corollary 1 — Putting $\alpha = 1$ in the above theorem we see that the radius of β-spiral convexity of β-spiral functions is given by

$$R(\alpha, \beta) = \frac{1}{(1 + 3 \cos^2 \beta)^{1/2} + \sqrt{3} \cos \beta}.$$

This result was obtained by Ruscheweyh and Singh (1976).

Corollary 2 — Taking $\beta = 0$, we see that every starlike function is α-convex in $| z | < [(1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2}]$, which is the result due to Miller *et al.* (1974).

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