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ON THE COEFFICIENTS OF A SUBCLASS OF STARLIKE FUNCTIONS

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Let $S^*(A, B)$ denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ regular in the unit disc $E = \{z : |z| < 1\}$ such that $f \in S^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in E$$

where A, B are arbitrary fixed numbers, $-1 \leqslant B < A \leqslant 1$.

In this paper, we obtain coefficient estimates for the class $S^*(A, B)$.

1. INTRODUCTION

Let U denote the class of functions $w(z) = \sum_{n=1}^{\infty} b_n z^n$ regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in E$. Let $S^*(A, B)$ denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ regular in E and such that $f \in S^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in E)$$

where A and B are arbitrary fixed numbers, $-1 < A \leqslant 1$, $-1 \leqslant B < 1$. It follows from definition of subordination that

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in U. \qquad \ldots(1.1)$$

The class $S^*(A, B)$ is a subclass of the class S^* of univalent starlike functions with respect to the origin.

The class $S^*(A, B)$ was studied by Janowski (1973) and he obtained the bounds for $|f(z)|$, $|f'(z)|$ and the exact value of radius of convexity for $S^*(A, B)$. However he did not consider the bounds for coefficients. It will be of interest to investigate the coefficient estimates for functions of the class $S^*(A, B)$. The purpose of this paper is to consider this problem.

$S^*(1, -1)$ coincides with S^*. By giving specific values to A and B we obtain the following important subclasses of starlike functions studied by various authors in earlier works:

(i) $S^*(1, 0) \equiv$ class of functions $f(z)$ satisfying the conditions

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in E)$$

studied by Singh (1968).

(ii) $S^*(\alpha, -\alpha) \equiv$ class of functions $f(z)$ satisfying the condition

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right)\Big/\left(\frac{zf'(z)}{f(z)} + 1\right)\right| < \alpha, \quad z \in E, \quad 0 < \alpha \leqslant 1$$

studied by Padmanabhan (1968).

(iii) $S^*\left(1, \frac{1}{\alpha} - 1\right) \equiv$ class of functions $f(z)$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)} - \alpha\right| < \alpha, \quad \alpha \geqslant \tfrac{1}{2}, \quad (z \in E)$$

studied by Singh and Singh (1974).

(iv) $S^*\left(\frac{b^2 - a^2 + a}{b}, \frac{1 - a}{b}\right) \equiv$ class of functions $f(z)$ satisfying the condition $\left|\frac{zf'(z)}{f(z)} - a\right| < b$, $(z \in E)$, $a + b \geqslant 1$, $b \leqslant a \leqslant b + 1$, studied by Silverman (1978).

Thus the results obtained here generalize the results due to Clunie and Keogh (1960), Pommerenke (1962), Singh (1968), Singh and Singh (1974) and Silverman (1978).

Our basic tool is the following lemma known as Rogosinski Inequality for majorization (Robertson 1967).

Lemma — Let $g(z) = \sum_{k=q}^{\infty} d_k z^k$ and $G(z) = \sum_{k=q}^{\infty} D_k z^k$, $q \geqslant 0$.

If $g(z) = w(z)\, G(z)$ where $w(z) \in U$, then

$$d_q = 0 \text{ and } \sum_{k=q+1}^{n} |d_k|^2, \leqslant \sum_{k=q}^{n-1} |D_k|^2, \quad (n = q + 1, q + 2, \ldots). \qquad \ldots(1.2)$$

2. Coefficient Inequalities

Theorem 1 — If $f \in S^*(A, B)$, then

$$|a_2| \leqslant (A - B); \qquad \ldots(2.1)$$

for $A - 2B \leqslant 1, n \geqslant 3$,

$$| a_n | \leqslant \frac{(A - B)}{n - 1}; \qquad \ldots(2.2)$$

for $A - (n - 1) B > (n - 2),\ n \geqslant 3$,

$$| a_n | \leqslant \frac{1}{(n - 1)!} \prod_{j=2}^{n} (A - (j - 1) B); \qquad \ldots(2.3)$$

and for $A - (n - 1) B < (n - 2), n \geqslant 4$,

$$| a_n | \leqslant \frac{1}{(n - 1)((n - 3)!)} \prod_{j=2}^{n-1} (A - (j - 1) B). \qquad \ldots(2.4)$$

Bounds (2.1), (2.2) and (2.3) are sharp.

PROOF : By (1.1) we have

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},\ w(z) \in U$$

or

$$\sum_{k=2}^{\infty} (k - 1) a_k z^k = w(z) \sum_{k=1}^{\infty} (A - Bk) a_k z^k.$$

On applying (1.2) we get

$$\sum_{k=2}^{n} (k - 1)^2 | a_k |^2 \leqslant \sum_{k=2}^{n-1} (A - Bk)^2 | a_k |^2,\ n = 2, 3, \ldots$$

or

$$| a_n |^2 \leqslant \frac{1}{(n - 1)^2} \left[(A - B)^2 + \sum_{k=2}^{n-1} ((A - kB)^2 - (k - 1)^2) | a_k |^2 \right]$$

$$(n = 2, 3, \ldots). \qquad \ldots(2.5)$$

Inequality (2.1) follows from (2.5).

$A - 2B \leqslant 1$ implies $A - (n - 1) B \leqslant (n - 2), (n \geqslant 3)$, and all the terms under the summation in (2.5) are non-positive and therefore we conclude that

$$| a_n | \leqslant \frac{(A - B)}{(n - 1)},\ A - 2B \leqslant 1,\ n \geqslant 3.$$

The equality signs in (2.1) and (2.2) are attained for the functions

$$f_n(z) = \begin{cases} z(1 + B\delta z^{n-1})^{(A-B)/(n-1)B}, \; B \neq 0 \\ z \exp\left(\dfrac{A\delta z^{n-1}}{n-1}\right), \; B = 0, \; |\delta| = 1. \end{cases} \qquad \ldots(2.6)$$

We now prove (2.3) when $A - (n-1)B > (n-2)$, $n \geqslant 3$. All the terms under the summation sign are positive. We prove the result by induction. Assuming (2.3) holds for $k = 3, 4, \ldots, (n-1)$. Then from (2.5) we obtain

$$|a_n|^2 \leqslant \frac{1}{(n-1)^2}\left[(A-B)^2 + \sum_{k=2}^{n-1} \frac{((A-kB)^2 - (k-1)^2)}{((k-1)!)^2} \times \prod_{j=2}^{k} (A-(j-1)B)^2\right]. \qquad \ldots(2.7)$$

In order to complete the proof it is sufficient to show that

$$\frac{1}{(m-1)^2}\left[(A-B)^2 + \sum_{k=2}^{m-1} ((A-kB)^2 - (k-1)^2)\,|a_k|^2\right]$$

$$= \frac{1}{((m-1)!)^2} \prod_{j=2}^{m} (A-(j-1)B)^2, \; m = 3, 4, \ldots \qquad \ldots(2.8)$$

where $A - (m-1)B > (m-2)$.

(2.8) is valid for $m = 3$. Let us assume that (2.8) is true for all m, $3 < m \leqslant n-1$. Then from (2.7), we get

$$|a_n| \leqslant \frac{1}{(n-1)^2}\left[(A-B)^2 + \sum_{k=2}^{n-1} ((A-kB)^2 - (k-1)^2)\,|a_k|^2\right]$$

$$= \frac{(n-2)^2}{(n-1)^2} \begin{cases} \dfrac{1}{(n-2)^2}\left\{(A-B)^2 + \displaystyle\sum_{k=2}^{n-2} \frac{(A-kB)^2 - (k-1)^2}{((k-1)!)^2} \times \prod_{j=2}^{k} (A-(j-1)B)^2\right\} \\ + \dfrac{1}{(n-2)^2} \dfrac{(A-(n-1)B)^2 - (n-2)^2}{((n-2)!)^2} \displaystyle\prod_{j=2}^{n-1} (A-(j-1)B)^2 \end{cases}$$

or

$$|a_n|^2 \leqslant \frac{(n-2)^2}{(n-1)^2}\left[\frac{1}{((n-2)!)^2} \prod_{j=2}^{n-1} (A-(j-1)B)^2\right] +$$

(*equation continued on p.* 638)

$$+ \frac{1}{((n-1)!)^2} [(A-(n-1)B)^2 - (n-2)^2]\left[\prod_{j=2}^{n-1} (A-(j-1)B)^2\right]$$

$$= \frac{1}{((n-1)!)^2} \prod_{j=2}^{n} (A-(j-1)B)^2.$$

The function

$$f_0(z) = z(1 + B\delta z)^{(A-B)/B} \; (|\delta| = 1) \qquad \ldots(2.9)$$

gives sharp estimates.

We now prove (2.4). If $A-(n-2)B > (n-3)$ and $A-(n-1)B \leqslant (n-2)$, $n \geqslant 4$, then because of (2.5) we have

$$|a_n|^2 \leqslant \frac{1}{(n-1)^2}\left[(A-B)^2 + \sum_{k=2}^{n-2} ((A-kB)^2 - (k-1)^2)\,|a_k|^2\right]. \qquad \ldots(2.10)$$

By the identity (2.8),

$$|a_n|^2 \leqslant \frac{1}{(n-1)^2}\left[\frac{(n-2)^2}{((n-2)!)^2} \prod_{j=2}^{n-1} (A-(j-1)B)^2\right], \; (n \geqslant 4)$$

or

$$|a_n| \leqslant \frac{1}{(n-1)((n-3)!)} \prod_{j=2}^{n-1} (A-(j-1)B).$$

Remark: For $1 < (A-2B) < 2+B$, (2.4) implies that

$$|a_4| \leqslant \tfrac{1}{3}(A-B)(A-2B)$$

which is not a sharp bound. We obtain exact bounds for a_4 when a_2, a_3 and a_4 are real.

Theorem 2 — Let $f \in S^*(A, B)$, and a_2, a_3 and a_4 are real, then

$$|a_4| \leqslant \begin{cases} \dfrac{(A-B)}{3}, & 1 \leqslant A-2B \leqslant \dfrac{4+B}{3}, \\[2ex] \dfrac{(A-B)}{3} + \dfrac{(3A-7B-4)^3(3A-7B+4)^3}{324(A-B)^3}, & \dfrac{4+B}{3} \leqslant (A-2B) \leqslant \dfrac{(4+B)}{3} + \dfrac{(A-B)^2}{2(3A-7B+4)}, \\[2ex] \dfrac{(A-B)(A-2B)(A-3B)}{3!}, & \dfrac{4+B}{3} + \dfrac{(A-B)^2}{2(3A-7B+4)} \leqslant (A-2B) \leqslant 2+B. \end{cases}$$

The results are sharp.

PROOF : By (1.1)

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \; w \in U. \qquad \text{...(2.11)}$$

Setting

$$w(z) = \frac{z(b_1 + \theta(z))}{1 + \bar{b}_1\theta(z)}, \; \theta(z) = \sum_{n=1}^{\infty} c_n z^n \in U. \qquad \text{...(2.12)}$$

After some computation, (2.11) yields

$$\sum_{k=2}^{\infty} [(k - 1)\, a_k - b_1(A - (k - 1)\, B)\, a_{k-1}]\, z^k$$

$$= \theta(z) \sum_{k=2}^{\infty} [(A - (k - 1)\, B)\, a_{k-1} - (k - 1)\, \bar{b}_1 a_k]\, z^k. \qquad \text{...(2.13)}$$

Equating the coefficients of z^2,

$$a_2 = (A - B)\, b_1 \text{ and therefore } \bar{a}_2 = (A - B)\, \bar{b}_1. \qquad \text{...(2.14)}$$

(2.13) can be written as

$$\sum_{k=3}^{\infty} [(k - 1)\, a_k - b_1(A - (k - 1)\, B)\, a_{k-1}]\, z^k$$

$$= \theta(z) \sum_{k=2}^{\infty} [(A - (k - 1)\, B)\, a_{k-1} - (k - 1)\, \bar{b}_1 a_k]\, z^k. \qquad \text{...(2.15)}$$

Putting $$\theta(z) = z\left[\frac{c_1 + \beta(z)}{1 + \bar{c}_1\beta(z)}\right], \; \beta \in U. \qquad \text{...(2.16)}$$

By (2.15) and (2.16) we have

$$\sum_{k=3}^{\infty} [\{(k - 1)\, a_k - b_1(A - (k - 1)\, B)\, a_{k-1}\}$$

$$- c_1\{(A - (k - 2)\, B)\, a_{k-2} - \bar{b}_1(k - 2)\, a_{k-1}\}]\, z^k$$

$$= \beta(z) \sum_{k=3}^{\infty} [\{(A - (k - 2)\, B)\, a_{k-2} - \bar{b}_1(k - 2)\, a_{k-1}\}$$

$$- \bar{c}_1\{(k - 1)\, a_k - b_1(A - (k - 1)\, B)\, a_{k-1}\}]\, z^k. \qquad \text{...(2.17)}$$

Equating the coefficients of z^3,

$$[2a_3 - b_1(A - 2B)\, a_2] = c_1[(A - B) - \bar{b}_1 a_2]. \qquad \text{...(2.18)}$$

By (2.14) and (2.18), we obtain

$$c_1 = \frac{2}{(A-B)}\left[a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right] \Big/ \left(1 - \frac{|a_2|^2}{(A-B)^2}\right). \qquad \ldots(2.19)$$

Now $|c_1| \leqslant 1$ and it follows from (2.19) that

$$\left|a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right| \leqslant \frac{(A-B)}{2}\left(1 - \frac{|a_2|^2}{(A-B)^2}\right). \qquad \ldots(2.20)$$

By (1.2), for $n = 4$, we obtain from (2.17)

$$|(3a_4 - b_1(A-3B)\,a_3) - c_1((A-2B)\,a_2 - 2\bar{b}_1 a_3)|$$
$$\leqslant |((A-B) - \bar{b}_1 a_2) - \bar{c}_1(2a_3 - b_1(A-2B)\,a_2)|\,. \qquad \ldots(2.21)$$

Using (2.14) and (2.19), (2.21) can be written as

$$\left|\left(3a_4 - \frac{(A-3B)\,a_2 a_3}{(A-B)}\right) - \frac{2}{(A-B)}\left((A-2B)\,a_2 - \frac{2\bar{a}_2 a_3}{(A-B)}\right)\right.$$
$$\left.\times\left(a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right)\Big/\left(1 - \frac{|a_2|^2}{(A-B)^2}\right)\right|$$
$$\leqslant (A-B)\Big/\left(1 - \frac{|a_2|^2}{(A-B)^2}\right)\left|\left(1 - \frac{|a_2|^2}{(A-B)^2}\right)^2 - \frac{4}{(A-B)^2}\right.$$
$$\left.\times\left|a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right|^2\right|$$

or

$$\left|3a_4 - \frac{(A-2B)(A-3B)\,a_2^3}{2(A-B)^2} - \frac{(3A-7B)\,a_2}{(A-B)}\left(a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right)\right.$$
$$\left.+ 4\frac{\bar{a}_2}{(A-B)^2}\left(a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right)^2\Big/\left(1 - \frac{|a_2|^2}{(A-B)^2}\right)\right|$$
$$\leqslant (A-B)\left(1 - \frac{|a_2|^2}{(A-B)^2}\right) - \frac{4}{(A-B)}$$
$$\times\left|a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)}\right|^2\Big/\left(1 - \frac{|a_2|^2}{(A-B)^2}\right). \qquad \ldots(2.22)$$

Assuming a_2, a_3 and a_4 to be real, (2.20) and (2.22) can be written as

$$a_3 - \frac{(A-2B)\,a_2^2}{2(A-B)} = \mu\frac{(A-B)}{2}\left[1 - \frac{a_2^2}{(A-B)^2}\right], \; (-1 \leqslant \mu \leqslant 1) \qquad \ldots(2.23)$$

and

$$3a_4 = \left[\begin{array}{l} \dfrac{(A-2B)(A-3B)a_2^3}{2(A-B)^2} + \mu\dfrac{(3A-7B)}{2}\left(1-\dfrac{a_2^2}{(A-B)^2}\right) \\ -\mu^2 a_2\left(1-\dfrac{a_2^2}{(A-B)^2}\right) + \lambda(A-B)(1-\mu^2)\left(1-\dfrac{a_2^2}{(A-B)^2}\right) \end{array}\right],$$

$$(-1 \leqslant \lambda \leqslant 1).$$

The right-hand side is maximum when $\lambda = 1$. Thus,

$$3a_4 \leqslant \frac{(A-2B)(A-3B)a_2^3}{2(A-B)^2}$$

$$+ (A-B)\left(1-\frac{a_2^2}{(A-B)^2}\right)\left[1-\mu^2-\frac{\mu^2 a_2}{(A-B)}+\frac{a_2(3A-7B)}{2(A-B)}\right]$$

$$= \frac{c(c-B)x^3}{2(c+B)^2} + (c+B)\left(1-\frac{x^2}{(c+B)^2}\right)\left[1-\mu^2-\frac{\mu^2 x}{(c+B)} + \frac{\mu x(3c-B)}{2(c+B)}\right]$$

$$= h(\mu, x), \text{ say, (where } A-2B = c \text{ and } a_2 = x). \qquad \ldots(2.24)$$

For extreme values, $\dfrac{\partial h}{\partial \mu} = 0 = \dfrac{\partial h}{\partial x}$ which yield

$$\mu = \frac{(3c-B)x}{4(c+B+x)} \qquad \ldots(2.25)$$

$$\left.\begin{array}{l} x = 0 \\ x = 2\dfrac{((3c-B)^2-16)}{3(c+B)} \end{array}\right]. \qquad \ldots(2.26)$$

It is easy to verify that $(0, 0)$ gives the maximum value of $h(\mu, x)$ provided $(3c-B) \leqslant 4$. From (2.24),

$$|a_4| \leqslant \frac{(A-B)}{3} \text{ provided } 1 \leqslant (A-B) \leqslant \frac{4+B}{3}. \qquad \ldots(2.27)$$

Also $\mu_0 = \dfrac{(3c-B)x_0}{4(c+B+x_0)}$, $x_0 = \dfrac{2((3c-B)^2-16)}{3(c+B)}$ gives maximum value of $h(\mu, x)$.

$$h(\mu_0, x_0) = (c+B) + \frac{1}{108}\,\frac{((3c-B)^2-16)^3}{(c+B)^3}$$

provided $3c - B \geqslant 4$ and $c \leqslant \dfrac{4+B}{3} + \dfrac{3(c+B)^2}{2(3c-B+4)}$.

From (2.24), we have

$$| a_4 | \leqslant \frac{(A - B)}{3} + \frac{1}{324} \frac{(3A - 7B - 4)^3 (3A - 7B + 4)^3}{(A - B)^3} \quad \ldots(2.28)$$

provided $\frac{4 + B}{3} \leqslant (A - 2B) \leqslant \frac{4 + B}{3} + \frac{3(A - B)^2}{2(3A - 7B + 4)}$.

When $x_0 \geqslant (A - B)$, maximum value of $h(\mu, x)$ occurs at $x_0 = (A - B)$. Thus, from (2.24) we obtain

$$| a_4 | \leqslant \frac{(A - B)(A - 2B)(A - 3B)}{3!} \text{ if } \frac{4 + B}{3} + \frac{3(A - B)^2}{2(3A - 7B + 4)} \leqslant (A - 2B) \leqslant 2 + B. \quad \ldots(2.29)$$

The bounds (2.28) and (2.29) coincide at $A - 2B = \frac{4 + B}{3} + \frac{3(A - B)^2}{2(3A - 7B + 4)}$.

Also the bound (2.27) and (2.28) coincide at $A - 2B = \frac{1}{3}(4 + B)$. Now we prove that bounds (2.27), (2.28) and (2.29) are attained by a function in $S^*(A, B)$. Equality sign in (2.27) is obtained for the function defined by (2.6) when $B \neq 0$ and $n = 4$. The function $f_0(z)$ given by (2.9) proves the statement regarding sharpness of (2.29).

From (2.11), by equating the coefficients we get

$$a_2 = (A - B) b_1 \quad \ldots(2.30)$$

$$2a_3 - a_2^2 = (A - B)(b_2 - Bb_1^2) \quad \ldots(2.31)$$

$$3a_4 - 3a_2a_3 + a_2^3 = (A - B)[b_3 - 2Bb_1b_2 + B^2b_1^3]. \quad \ldots(2.32)$$

Setting

$$w(z) = z\left[b_1 + z\left(\frac{\mu_0 + z}{1 + \mu_0 z}\right)\right] \Big/ \left[1 + b_1 z\left(\frac{\mu_0 + z}{1 + \mu_0 z}\right)\right]$$

where $\mu_0 = \frac{(3c - B) x_0}{4(c + B + x_0)}$.

Then

$$b_2 = (1 - b_1^2), \quad \ldots(2.33)$$

$$b_3 = 1 - b_1^2 - \frac{b_2^2}{(1 - b_1)}. \quad \ldots(2.34)$$

From (2.30), (2.31) and (2.33) we get

$$a_3 = \frac{(A - 2B) a_2^2}{2(A - B)} + \mu_0 \frac{(A - B)}{2}\left[1 - \frac{a_2^2}{(A - B)^2}\right] \quad \ldots(2.35)$$

which is the result (2.23) with μ replaced by μ_0.

Because of (2.30), (2.32), (2.33), (2.34) and (2.35),

$$3a_4 = \left[\frac{3(A-2B)}{2(A-B)} - 1 + \frac{B^2}{(A-B)^2}\right] a_2^3$$

$$+ 3a_0 a_2 \frac{(A-B)}{2}\left(1 - \frac{a_2^2}{(A-B)^2}\right)$$

$$+ (A-B)\left[1 - \frac{a_2^2}{(A-B)^2} - \mu_0^2\left(1 - \frac{a_2^2}{(A-B)^2}\right)\left(1 + \frac{a_2}{(A-B)}\right)\right.$$

$$\left. - \frac{2B\mu_0 a_2}{(A-B)}\left(1 - \frac{a_2^2}{(A-B)^2}\right)\right]$$

$$= \frac{(A-2B)(A-3B)\,a_2^3}{2(A-B)^2} + (A-B)\left(1 - \frac{a_2^2}{(A-B)^2}\right)$$

$$\times\left[1 - \mu_0^2 - \frac{\mu_0^2 a_2}{(A-B)} + \frac{(3A-7B)\,\mu_0 a_2}{2(A-B)}\right].$$

This is the result (2.24) with μ replaced by μ_0.

Remark : In the general case when a_2, a_3 and a_4 are complex we have not been able to obtain sharp results for a_4. However we obtain rough estimates given below.

Theorem 3 — Let $f \in S^*(A, B)$, then

$$|a_4| \leqslant \begin{cases} \dfrac{(A-B)}{3}, & 1 \leqslant (A-2B) \leqslant \dfrac{B + 2(10-B^2)^{1/2}}{5}, \\[2ex] \dfrac{(A-B)^2\,(3A-7B)}{12\left[((A-2B)^2-1)\,(4-(A-3B)^2)\right]^{1/2}}, & \tfrac{1}{5}(B + 2(10-B^2)^{1/2}) \leqslant (A-2B) \leqslant c_0, \\[2ex] \dfrac{(A-B)\,(A-2B)\,(A-3B)}{3!}, & c_0 \leqslant (A-2B) \leqslant 2 + B \end{cases}$$

where c_0 is the least positive root of the equation

$$2c^4 - 4Bc^3 - (5 - 2B^2)\,c^2 + 2Bc - B^2 = 0, \quad c = (A-2B).$$

PROOF : From (2.15) by applying (1.2) wet get

$$\sum_{k=3}^{n} |\,(k-1)\,a_k - b_1(A - (k-1)\,B)\,a_{k-1}\,|^2$$

$$\leqslant \sum_{k=2}^{n-1} |\,(A - (k-1)\,B)\,a_{k-1} - (k-1)\,\bar{b}_1 a_k\,|^2, \quad b_1 = \frac{a_2}{(A-B)}$$

or

$$\left| (n-1)\, a_n - \frac{(A-(n-1)\,B)\, a_2 a_{n-1}}{(A-B)} \right|^2$$

$$\leqslant \sum_{k=2}^{n-1} \left| (A-(k-1)\,B)\, a_{k-1} - \frac{(k-1)\, \bar{a}_2 a_k}{(A-B)} \right|^2$$

$$- \sum_{k=3}^{n-1} \left| (k-1)\, a_k - \frac{(A-(k-1)\,B)\, a_2 a_{k-1}}{(A-B)} \right|^2$$

$$= \left(1 - \frac{|a_2|^2}{(A-B)^2}\right) \sum_{k=2}^{n-1} [(A-(k-1)\,B)^2\, |a_{k-1}|^2 - (k-1)^2\, |a_k|^2],$$

$$(n = 3, 4, \ldots) \quad \ldots(2.36)$$

For $n = 3$, we have

$$\left| a_3 - \frac{(A-2B)\, a_2^2}{2(A-B)} \right| \leqslant \frac{(A-B)}{2} \left(1 - \frac{|a_2|^2}{(A-B)^2}\right). \quad \ldots(2.37)$$

If we take $n = 4$ in (2.36), we get

$$\left| 3a_4 - \frac{(A-3B)\, a_2 a_3}{(A-B)} \right| \leqslant \left[\left(1 - \frac{|a_2|^2}{(A-B)^2}\right) \times \{(A-B)^2 + ((A-2B)^2 - 1)\, |a_2|^2 - 4\, |a_3|^2\}\right]^{1/2}.$$

Since $(A - 3B) > 0$, it follows that

$$3\, |a_4| \leqslant \frac{(A-3B)\, |a_2|\ \ |a_3|}{(A-B)} + \left[\left(1 - \frac{|a_2|^2}{(A-B)^2}\right) \times \{(A-B)^2 + ((A-2B)^2 - 1)\, |a_2|^2 - 4\, |a_3|^2\}\right]^{1/2}. \quad \ldots(2.38)$$

Putting $|a_3| = y$ and denoting the right-hand side of (2.38) by $H(y)$ we have

$$H(y) = \frac{(A-3B)\, |a_2|\, y}{(A-B)} + \left[\left(1 - \frac{|a_2|^2}{(A-B)^2}\right) \times \{(A-B)^2 + ((A-2B)^2 - 1)\, |a_2|^2 - 4y^2\}\right]^{1/2}. \quad \ldots(2.39)$$

and

$$H'(y) = \frac{(A-3B)\, |a_2|}{(A-B)} - \frac{4y\, \{1 - (|a_2|^2/(A-B)^2)\}^{1/2}}{[(A-B)^2 + ((A-2B)^2 - 1)\, |a_2|^2 - 4y^2]^{1/2}}. \quad \ldots(2.40)$$

An easy computation shows that $H(y)$ is maximum at

$$y = | a_3 | = \frac{(A - 3B) | a_2 |}{2(A - B)} \times \left[\frac{(A - B)^2 + ((A - 2B)^2 - 1) | a_2 |^2}{4 - (| a_2 |^2/(A - B)^2)(4 - (A - 3B)^2)}\right]^{1/2}. \quad ...(2.41)$$

The value of a_3 is admissible if it satisfies the inequality (2.37). That is if,

$$\frac{(A - 3B) | a_2 |}{2(A - B)} \left[\frac{(A - B)^2 + ((A - 2B)^2 - 1) | a_2 |^2}{4 - (| a_2 |^2/(A - B)^2)(4 - (A - 3B)^2)}\right]^{1/2} \leqslant \frac{(A - 2B) | a_2 |^2}{2(A - B)} + \frac{(A - B)}{2}\left(1 - \frac{| a_2 |^2}{(A - B)^2}\right). \quad ...(2.42)$$

Putting $c = (A - 2B) > 1$, $t = \frac{| a_2 |^2}{(A - B)^2}$, $0 \leqslant t \leqslant 1$, (2.42) can be written as

$$(c - B)^2 t \left[\frac{1 + (c^2 - 1) t}{4 - (4 - (c - B)^2) t}\right] \leqslant [1 + (c - 1) t]^2$$

or

$$(t - 1) [(c - 1)^2 (4 - (c - B)^2) t^2 + 8(c - 1) t + 4] \leqslant 0.$$

Now $(c - 1)^2 (4 - (c - B)^2) t^2 + 8(c - 1) t + 4 > 0.$

Thus, $t - 1 \leqslant 0$ which is true.

From (2.38),

$$\begin{aligned} 3 | a_4 | &\leqslant \text{Max } H(y) \\ &= (A - 3B) | a_2 | y + 4y \frac{\{1 - (| a_2 |^2/(A - B)^2)\}}{\{(A - 3B)/(A - B)\} | a_2 |} \\ &= \frac{(A - B)}{2} [\{1 + ((A - 2B)^2 - 1) s^2\}\{4 - (4 - (A - 3B)^2) s^2\}]^{1/2}, \\ &\qquad \left(s = \frac{| a_2 |}{(A - B)} \leqslant 1\right) \\ &= \frac{(A - B)}{2} [F(y)]^{1/2}, \end{aligned} \quad ...(2.43)$$

where

$$F(s) = [1 + ((A - 2B)^2 - 1) s^2] [4 - (4 - (A - 3B)^2) s^2]. \quad ...(2.44)$$

On differentiation,

$$F'(s) = 2s [\{4(A - 2B)^2 + (A - 3B)^2 - 8\} - 2s^2((A - 2B)^2 - 1) (4 - (A - 3B)^2)]. \quad ...(2.45)$$

For extreme values $F'(s) = 0$ which yields

$$s = 0,\ s^2 = s_1^2 = \frac{4(A - 2B)^2 + (A - 3B)^2 - 8}{2((A - 2B)^2 - 1)(4 - (A - 3B)^2}. \qquad \ldots(2.46)$$

It can be easily shown that $s = 0$ gives maximum value of $F(s)$ provided

$$c \leqslant \frac{B + 2(10 - B^2)^{1/2}}{5}.$$

Thus from (2.43), we obtain

$$|a_4| \leqslant \frac{(A - B)}{3} \quad \text{if} \quad 1 \leqslant (A - 2B) \leqslant \frac{B + 2(10 - B^2)^{1/2}}{5}. \qquad \ldots(2.47)$$

$s_1^2 < 1$ implies that

$$p(c) = 2c^4 - 4Bc^3 - (5 - 2B^2)c^2 + 2Bc - B^2 < 0$$

$$p(1) = (B + 1)(B - 3) < 0$$

$$p(2 + B) = 4(2 + B)^2 - 4 > 0.$$

We conclude that $p(c) = 0$ has a real positive root c_0 between 1 and $(2 + B)$.

Also, $s^2 = s_1^2$ gives maximum value of $F(s)$ provided

$$\frac{B + 2(10 - B^2)^{1/2}}{5} \leqslant (A - 2B) \leqslant c_0.$$

From (2.43), we obtain

$$|a_4| \leqslant \frac{(A - B)^2(3A - 7B)}{12\,[((A - 2B)^2 - 1)(4 - (A - 3B)^2)]^{1/2}} \qquad \ldots(2.48)$$

if
$$\frac{B + 2(10 - B^2)^{1/2}}{5} \leqslant (A - 2B) \leqslant c_0.$$

When $s^2 \geqslant 1$, $F(s)$ will attain its maximum values at $s^2 = 1$. In this situation,

$$|a_4| \leqslant \frac{(A - B)(A - 2B)(A - 3B)}{3!},\ c_0 \leqslant (A - 2B) \leqslant 2 + B. \ldots(2.49)$$

Bounds (2.47) and (2.48) coincide if

$$25c^4 - 20Bc^3 + (14B^2 - 80)c^2 - 4B(B^2 - 8)c + (B^2 - 8)^2 = 0$$

Or

$$[5c^2 - 2Bc + (B^2 - 8)]^2 = 0$$

Or
$$c = \frac{B + 2(10 - B^2)^{1/2}}{5}.$$

Bounds (2.48) and (2.49) are equal if

$$4c^8 - 16Bc^7 + (24B^2 - 20)\, c^6 + (48B - 16B^2)\, c^5 + (4B^4 - 40B^2 + 25)\, c^4 + (16B^2 - 20B)\, c^3 + (14B - 4B^4)\, c^2 - 4B^3c + B^4 = 0$$

or if

$$[2c^4 - 4Bc^3 - (5 - 2B^2)\, c^2 + 2Bc - B^2]^2 = 0.$$

That is, c_0 is the least positive root of the equation

$$2c^4 - 4B^4c^3 - (5 - 2B^2)\, c^2 + 2Bc - B^2 = 0.$$

Corollary 3.1 (Clunie and Keogh 1960, Pommerenke 1962) — Let $f \in S^*$, then from (2.36) we have

$$\left| (n - 1)\, a_n - \frac{n}{2}\, a_2 a_{n-1} \right|^2 \leqslant \left[1 - \frac{|\, a_2 \,|^2}{4} \right] \times \left[4 + 4 \sum_{k=2}^{n-2} k \,|\, a_k \,|^2 - (n - 2)^2 \,|\, a_{n-1} \,|^2 \right].$$

Corollary 3.2 (Singh and Singh 1974) — Let $f \in S^*(\alpha)$, then from (2.36) we have

$$\left| (n - 1)\, a_n - \frac{(n\alpha - n + 1)}{2\alpha - 1}\, a_2 a_{n-1} \right|^2$$

$$\leqslant \left[1 - \frac{\alpha^2 \,|\, a_2 \,|^2}{(2\alpha - 1)^2} \right] \left[\frac{(2\alpha - 1)^2}{\alpha^2} + \frac{(2\alpha - 1)}{\alpha^2} \sum_{k=2}^{n-2} k(2\alpha - k) \,|\, a_k \,|^2 - (n - 2)^2 \,|\, a_{n-1} \,|^2 \right].$$

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