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ON A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Let $S^*(A, B)$ denote the class of functions $g(z) = z + b_2z^2 + b_3z^3 + \ldots$ regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition

$$\frac{zg'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in E, \; -1 \leqslant B < A \leqslant 1.$$

Let $C^*(A, B)$ denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ regular in E, and satisfying the condition $\text{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \; z \in E.$

In this paper we determine the coefficient estimates, distortion theorems and radius of convexity for the class $C^*(A, B)$.

1. INTRODUCTION

Let $\mathcal{P}$ be the class of functions

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \qquad \ldots(1)$$

regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition Re $P(z) > 0$.

Let U be the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \qquad \ldots(2)$$

regular in E and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1, \; z \in E$. Let us denote by S the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \ldots(3)$$

regular and univalent in E.

Let $S^*(A, B)$ be the class of functions

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad \ldots(4)$$

regular in E and satisfying the condition

$$\frac{zg'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in E, \; -1 \leqslant B < A \leqslant 1.$$

By definition of subordination it follows that $g(z) \in S^*(A, B)$ has a representation of the form

$$\frac{zg'(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \; w(z) \in U. \qquad \ldots(5)$$

Obviously $S^*(A, B)$ is a subclass of the class S^* of starlike functions with respect to the origin.

Let $C^*(A, B)$ be the class of functions of the form (3) and satisfying the condition

$$\text{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad \ldots(6)$$

where $g(z) \in S^*(A, B)$. Then $C^*(A, B)$ is a subclass of the class C of close-to-convex functions.

Let $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} \beta_n z^n$ be any two functions, then by $\phi(z) * \psi(z)$ we shall mean the Hadamard product or convolution of $\phi(z)$ and $\psi(z)$, that is

$$\phi(z) * \psi(z) = \sum_{n=0}^{\infty} \alpha_n \beta_n z^n.$$

In the present paper we shall study the class $C^*(A, B)$ and obtain coefficient estimates, distortion theorems and radius of convexity. Taking $A = 1$ and $B = -1$ we get well-known results for the class C.

§2. We need the following lemmas:

Lemma 1 — If $g \in S^*(A, B)$, then

$$| b_2 | \leqslant (A - B) \qquad \ldots(7)$$

$$| b_n | \leqslant \frac{1}{(n-1)!} \prod_{j=2}^{n} (A - (j-1) B), \; A - (n-1) B \geqslant (n-2), \; n \geqslant 3. \qquad \ldots(8)$$

Equality sign in (7) is attained by the function $g_0(z)$ defined by

$$g_0(z) = \begin{cases} z(1 + B\delta z)^{(A-B)/B}, & B \neq 0 \\ z \exp (A\delta z), & B = 0, \; | \delta | = 1. \end{cases} \qquad \ldots(9)$$

Equality sign in (8) is attained by the function $g_0(z)$ defined above when $B \neq 0$. Goel and Mehrok (1981) proved the above results.

Lemma 2 — If $\psi(z)$ is regular in E, $\phi(z)$ and $h(z)$ are convex univalent in E such that $\psi(z) \prec \phi(z)$, then $\psi(z) * h(z) \prec \phi(z) * h(z)$, $z \in E$. This result is due to Ruscheweyh and Sheil-Small (1973).

Lemma 3 — If $g \in S^*(A, B)$, then for all $|s| \leqslant 1$, $|t| \leqslant 1$, $(s \neq t)$

$$\frac{tg(sz)}{sg(tz)} \prec \left[\frac{(1 + Bsz)}{1 + Btz}\right]^{(A-B)/B}, \; B \neq 0; \qquad \ldots(10)$$

$$\frac{tg(sz)}{sg(tz)} \prec \exp A(s - t)\, z, \; B = 0. \qquad \ldots(11)$$

PROOF : The proof is similar to the one given by Ruscheweyh (1976).

We first consider the case when $B \neq 0$. We have

$\dfrac{zg'(z)}{g(z)} \prec \dfrac{1 + Az}{1 + Bz}$, $z \in E$. This implies that

$$\frac{zg'(z)}{g(z)} - 1 \prec \frac{1 + Az}{1 + Bz} - 1 = \frac{(A - B)\, z}{1 + Bz}, \qquad \ldots(12)$$

where $\dfrac{(A - B)\, z}{1 + Bz}$ is convex, univalent in E. For $|s| \leqslant 1$, $|t| \leqslant 1$, $(s \neq t)$,

$$h(z) = \int_0^z \left(\frac{s}{1 - su} - \frac{t}{1 - tu}\right) du \qquad \ldots(13)$$

is convex, univalent in E. (12) and (13) satisfy the conditions of Lemma 2, and therefore

$$\left[\frac{zg'(z)}{g(z)} - 1\right] * h(z) \prec \frac{(A - B)\, z}{(1 + Bz)} * h(z). \qquad \ldots(14)$$

Now for every regular function $p(z)$ will $p(0) = 0$, we have

$$p(z) * h(z) = \int_{tz}^{sz} p(u) \frac{du}{u}. \qquad \ldots(15)$$

By the application of (15), (14) can be written as

$$\int_{tz}^{sz} \left[\frac{ug'(u)}{g(u)} - 1\right] \frac{du}{u} \prec (A - B) \int_{tz}^{sz} \frac{du}{1 + Bu}$$ from which (10) follows.

Similarly for $B = 0$, we obtain (11).

Lemma 4 — If $g \in S^*(A, B)$, then for $|z| = r < 1$

$$r(1 - Br)^{(A-B)/B} \leqslant |g(z)| \leqslant r(1 + Br)^{(A-B)/B}, \; B \neq 0 \qquad \ldots(16)$$

$$r \exp(-Ar) \leqslant |g(z)| \leqslant r \exp(Ar), \; B = 0. \qquad \ldots(17)$$

$$\left|\arg \frac{g(z)}{z}\right| \leqslant \frac{(A-B)}{B} \sin^{-1}(Br), \ B \neq 0 \qquad \ldots(18)$$

$$\left|\arg \frac{g(z)}{z}\right| \leqslant Ar, \ B = 0. \qquad \ldots(19)$$

These bounds are sharp, being attained by the function $g_0(z)$ defined by (9).

PROOF : Janowski (1973) proved the results (16) and (17) by means of variational methods. However, we deduce them from Lemma 3.

Taking $s = 1$, $t = 0$ in (10) and (11), we get

$$\frac{g(z)}{z} \prec (1 + Bz)^{(A-B)/B}, \ B \neq 0; \qquad \ldots(20)$$

$$\frac{g(z)}{z} \prec \exp(Az), \ B = 0. \qquad \ldots(21)$$

(20) implies that $\frac{g(z)}{z} = (1 + Bw(z))^{(A-B)/B}$, $B \neq 0$.

(i) When $B > 0$.

$$\left|\frac{g(z)}{z}\right| = |(1 + Bw(z))^{(A-B)/B}| = \left|\exp\left[\frac{(A-B)}{B}\log(1 + Bw(z))\right]\right|$$

$$= \exp \operatorname{Re}\left[\frac{(A-B)}{B} \cdot \log(1 + Bw(z))\right]$$

$$= \exp\left(\frac{(A-B)}{B}\log|1 + Bw(z)|\right)$$

$$\leqslant |1 + Bw(z)|^{(A-B)/B} \leqslant (1 + Br)^{(A-B)/B}.$$

(ii) When $B < 0$, put $B = -C$, $C > 0$.

$$|(1 + Bw(z))^{(A-B)/B}| = |((1 - Cw(z))^{-1})^{(A-B)/C}|$$

$$= |(1 - Cw(z))^{-1}|^{(A-B)/C}$$

$$\leqslant \left(\frac{1}{1 - Cr}\right)^{(A-B)/C}$$

$$= (1 + Br)^{(A-B)/B}.$$

Similarly (17) is a direct consequence of (21). For $|z| = r$, from (20), we get

$$\left|\arg \frac{g(z)}{z}\right| = \frac{(A-B)}{B} |\arg(1 + Bw(z))|$$

$$\leqslant \frac{(A-B)}{B} \sin^{-1}(Br).$$

Similarly (19) is a direct consequence of (21).

3. Coefficient Inequalities

Theorem 1 — If $f \in C^*(A, B)$ then for $A - (n-1)B \geqslant (n-2)$, $n \geqslant 2$,

$$|a_n| \leqslant \frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B) + \frac{2}{n}\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]. \qquad \ldots(22)$$

The result is sharp.

Proof ; Since $f \in C^*(A, B)$, it follows that

$$zf'(z) = g(z)\,P(z) \qquad \ldots(23)$$

where $g \in S^*(A, B)$ and $P \in \mathcal{P}$.

On equating the coefficients of z^n, we get

$$na_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + \ldots + p_{n-2} b_2 + p_{n-1}.$$

This gives

$$n|a_n| \leqslant |b_n| + |p_1|\,|b_{n-1}| + |p_2|\,|b_{n-2}| + \ldots + |p_{n-2}|\,|b_2| + |p_{n-1}|.$$

It is well known that $|p_n| \leqslant 2$ for $n \geqslant 1$. Thus

$$n|a_n| \leqslant |b_n| + 2[|b_{n-1}| + |b_{n-2}| + \ldots + |b_2| + 1]. \qquad \ldots(24)$$

Substituting the value from (8) in (24), we obtain

$$n|a_n| \leqslant \frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B) + 2\left[\frac{1}{(n-2)!}\prod_{j=2}^{n-1}(A-(j-1)B) + \frac{1}{(n-3)!}\prod_{j=2}^{n-2}(A-(j-1)B) + \ldots + (A-B) + 1\right]$$

which, on simplification, takes the form of (22). The function $f_0(z)$ defined by

$$f_0'(z) = \begin{cases}\left(\dfrac{1+\delta_1 z}{1-\delta_1 z}\right)[1+B\delta_2 z]^{(A-B)/B}, & B \neq 0; \\ \left(\dfrac{1+\delta_1 z}{1-\delta_1 z}\right)\exp(A\delta_2 z), & B = 0,\ |\delta_1| = |\delta_2| = 1\end{cases} \qquad \ldots(25)$$

shows that the bound (22) is sharp for each $n \geqslant 2$.

Let $A = 1$, $B = -1$ in Theorem 1, we get the following:

Corollary — $$| a_n | \leqslant \frac{1}{n!} \prod_{j=2}^{n} (j) + \frac{2}{n}\left[1 + \sum_{k=2}^{n-1} \frac{1}{(k-1)!} \prod_{j=2}^{k} (j)\right]$$

$$= 1 + \frac{2}{n} \times \frac{n(n-1)}{2} = n,$$

a result which was established earlier by Reade (1955).

Remark : For $A - (n-1) B < (n-2)$, we have been able to find the estimate on $| a_3 |$ only.

Theorem 2 — If $f \in C^*(A, B)$, then

$$| a_3 | \leqslant \begin{cases} \frac{2}{3}(A - B + 1) - \dfrac{(A-B)(A-2B)}{6}, & -B \leqslant (A - 2B) \leqslant 0: \\ \frac{2}{3}(A - B + 1) + \dfrac{(A-B)(A-2B)}{6}, & 0 \leqslant (A - 2B) \leqslant 1. \end{cases} \quad \ldots(26)$$

PROOF : From (5) equating the coefficient of the like powers of z,

$$b_2 = (A - B)\, c_1 \quad \ldots(27)$$

$$2b_3 = (A - B)\, [c_2 + (A - 2B)\, c_1^2]. \quad \ldots(28)$$

It is well known that if $w(z) \in U$, then $| c_2 | \leqslant 1 - | c_1 |^2$ which along with (27) and (28) yields

$$\left| b_3 - \frac{(A-2B)\, b_2^2}{2(A-B)} \right| \leqslant \frac{(A-B)}{2}\left[1 - \frac{| b_2 |^2}{(A-B)^2}\right]. \quad \ldots(29)$$

From (29) it follows that

$$b_3 = \frac{(A-2B)\, b_2^2}{2(A-B)} + \delta\, \frac{(A-B)}{2}\left[1 - \frac{| b_2^2 |}{(A-B)^2}\right], \; | \delta | \leqslant 1.$$

Putting $b_3 = | b_3 | e^{i\alpha}$, $\delta = | \delta | e^{i\beta}$, $0 \leqslant \alpha \leqslant 2\pi$, $0 \leqslant \beta \leqslant 2\pi$, we have

$$| b_3 |^2 = \frac{(A-2B)^2 | b_2 |^4}{4(A-B)^2} + | \delta |^2 \frac{(A-B)^2}{4}\left[1 - \frac{| b_2 |^2}{(A-B)^2}\right]^2$$
$$+ \frac{(A-2B)}{2} | \delta | \; | b_2 |^2 \left[1 - \frac{| b_2 |^2}{(A-B)^2}\right] \cos (2\alpha + \beta).$$

Consider the case when $-B \leqslant (A - 2B) \leqslant 0$. $| b_3 |^2$ attains its maximum value at $2\alpha + \beta = \pi$. In this case,

$$| b_3 | \leqslant \frac{(A-B)}{2} - \left[\frac{(1 + (A-2B))}{2(A-B)}\right] | b_2 |^2. \quad \ldots(30)$$

For $n = 3$, (24) gives

$$3 \mid a_3 \mid \leqslant \mid b_3 \mid + 2 \mid b_2 \mid + 2 \quad \ldots(31)$$

which together with (30) yields

$$3 \mid a_3 \mid \leqslant \frac{(A-B)}{2} - \frac{(1+(A-2B))}{2(A-B)} x^2 + 2x + 1, \mid b_2 \mid = x$$

$$= H(x), \text{ say.}$$

$$H'(x) = 2 - \frac{(1+(A-2B))}{(A-B)} x, \; 0 \leqslant x \leqslant (A-B).$$

Since $x = \mid b_2 \mid \leqslant (A-B)$, it is clear that $H'(x) > 0$ for $-B \leqslant (A-2B) \leqslant 0$. $H(x)$ attains its maximum value at $x = (A-B)$.

$$3 \mid a_3 \mid \leqslant 2 + 2(A-B) + \frac{(A-B)}{2} - [1 + (A-2B)] \times \frac{(A-B)}{2}$$

or

$$\mid a_3 \mid \leqslant \tfrac{2}{3}((A-B)+1) - \frac{(A-B)(A-2B)}{6}. \quad \ldots(32)$$

For $0 \leqslant (A-2B) \leqslant 1$, we have

$$\mid b_3 \mid \leqslant \frac{(A-B)}{2} - \frac{(1-(A-2B))}{2(A-B)} \mid b_2 \mid^2. \quad \ldots(33)$$

From (31) and (33) we have

$$3 \mid a_3 \mid \leqslant \frac{(A-B)}{2} - \frac{(1-(A-2B))}{2(A-B)} x^2 + 2x + 2$$

$$= F(x), \mid b_2 \mid = x, \; 0 \leqslant x \leqslant (A-B).$$

$$F'(x) = 2 - \frac{(1-(A-2B))}{(A-B)} x > 0 \quad \text{for} \quad 0 \leqslant (A-2B) \leqslant 1.$$

Maximum value of $F(x)$ occurs at $x = (A-B)$. Hence

$$\mid a_3 \mid \leqslant \tfrac{2}{3} [(A-B)+1] + \frac{(A-B)(A-2B)}{6}. \quad \ldots(34)$$

The bound (34) is exact, being attained by the function $f_0(z)$ defined by (25).

4. Distortion Theorems

Theorem 3 — If $f \in C^*(A, B)$, then for $\mid z \mid = r < 1$

$$\left(\frac{1-r}{1+r}\right)(1-Br)^{(A-B)/B} \leqslant \mid f'(z) \mid \leqslant \left(\frac{1+r}{1-r}\right)(1+Br)^{(A-B)/B}, \; B \neq 0 \quad \ldots(35)$$

$$\left(\frac{1-r}{1+r}\right) \exp(-Ar) \leqslant | f'(z) | \leqslant \left(\frac{1+r}{1-r}\right) \exp(Ar),\ B = 0; \qquad \ldots(36)$$

$$\int_0^r \left(\frac{1-r}{1+r}\right) (1 - Br)^{(A-B)/B}\, dr \leqslant | f(z) |$$

$$\leqslant \int_0^r \left(\frac{1+r}{1-r}\right) (1 + Br)^{(A-B)/B}\, dr,\ B \neq 0; \qquad \ldots(37)$$

$$\int_0^r \left(\frac{1-r}{1+r}\right) \exp(-Ar)\, dr \leqslant | f(z) | \leqslant \int_0^r \left(\frac{1+r}{1-r}\right) \exp(Ar)\, dr,\ B = 0. \qquad \ldots(38)$$

All these inequalities are sharp.

PROOF : From (23) we have

$$| f'(z) | = \left|\frac{g(z)}{z}\right| \, | P(z) | . \qquad \ldots(39)$$

It is well known that

$$\frac{1-r}{1+r} \leqslant | P(z) | \leqslant \frac{1+r}{1-r}. \qquad \ldots(40)$$

Using (16), (17) and (40) in (39), we obtain (35) and (36).

Now

$$| f(z) | = \left| \int_0^r f'(z)\, dz \right|$$

$$\leqslant \int_0^{|z|} | f'(z) |\, dr$$

$$\leqslant \begin{cases} \displaystyle\int_0^r \left(\frac{1+r}{1-r}\right) (1 + Br)^{(A-B)/B}\, dr, & B \neq 0; \\ \displaystyle\int_0^r \left(\frac{1+r}{1-r}\right) \exp(Ar)\, dr, & B = 0. \end{cases}$$

Let z_0, $| z_0 | = r$, be so chosen that $| f(z_0) | \leqslant | f(z) |$ for all z, $| z | = r$. If $L(z_0)$ is the pre-image of the segment $[0, \overline{f(z_0)}]$ in E then $| f(z_0) | = \int_{L(z_0)} | f'(z) |\, dr$.

$$\geqslant \begin{cases} \int_0^r \left(\frac{1-r}{1+r}\right)(1-Br)^{(A-B)/B}\, dr, & B \neq 0; \\ \int_0^r \left(\frac{1-r}{1+r}\right)\exp(-Ar)\, dr, & B = 0. \end{cases}$$

Equality signs in (35), (36), (37) and (38) are attained by the function defined by (25).

Corollary — Let $A = 1$, $B = -1$ in Theorem 3. Then we obtain well-known distortion theorems for close-to-convex functions.

5. Argument of $f'(z)$

Theorem 4 — If $f \in C^*(A, B)$, then

$$|\arg f'(z)| \leqslant \frac{(A-B)}{B}\sin^{-1}(Br) + \sin^{-1}\left(\frac{2r}{1+r^2}\right),\ B \neq 0 \qquad \ldots(41)$$

$$|\arg f'(z)| \leqslant Ar + \sin^{-1}\left(\frac{2r}{1+r^2}\right),\ B = 0. \qquad \ldots(42)$$

These inequalities are sharp.

Proof: From (23) we have $f'(z) = \frac{g(z)}{z} P(z)$.

Thus

$$|\arg f'(z)| \leqslant \left|\arg \frac{g(z)}{z}\right| + |\arg P(z)|. \qquad \ldots(43)$$

Also

$$|\arg P(z)| \leqslant \sin^{-1}\left(\frac{2r}{1+r^2}\right). \qquad \ldots(44)$$

(43) and (44) together with (18) and (19) yield (41) and (42). Equality signs in (41) and (42) hold for the function $f_1(z)$ and $f_2(z)$ respectively, where

$$f_1'(z) = \frac{1+\delta_1 z}{1-\delta_1 z}(1 + B\delta_2 z)^{(A-B)/B}$$

$$f_2'(z) = \frac{1+\delta_1 z}{1-\delta_1 z}\exp(A\delta_2 z)$$

where

$$\delta_1 = \frac{ir}{z},\ \delta_2 = \frac{r}{z}[-Br + i(1 - B^2r^2)^{1/2}].$$

Taking $A = 1$, $B = -1$ in the above theorem we have

Corollary — $|\arg f'(z)| \leqslant 2\sin^{-1}(r) + \sin^{-1}\left(\frac{2r}{1+r^2}\right)$ which is the result proved by Ogawa (1959) and Krzyz (1963).

6. Radius of Convexity for $C^*(A, B)$

Theorem 5 — If $f \in C^*(A, B)$ then it maps the disc $|z| \leqslant r_0$ onto a convex domain, where r_0 is the smallest positive root of the equation

$$Ar^3 - (1-2B)r^2 - (2+A)r + 1 = 0. \qquad \text{...(45)}$$

The number r_0 cannot be replaced by any larger one.

Proof : From (23), on differentiation logarithmically we have

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{zg'(z)}{g(z)} + \frac{zP'(z)}{P(z)}.$$

Thus

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) - \left|\frac{zP'(z)}{P(z)}\right|. \qquad \text{...(46)}$$

From (5), for $|w(z)| \leqslant r$, it is readily verified that

$$\frac{1-Ar}{1-Br} \leqslant \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \leqslant \frac{1+Ar}{1+Br}. \qquad \text{...(47)}$$

It is well known that

$$\left|\frac{zP'(z)}{P(z)}\right| \leqslant \frac{2r}{1-r^2}. \qquad \text{...(48)}$$

From (46), (47) and (48) we have

$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1-Ar}{1-Br} - \frac{2r}{1-r^2} > 0$, provided $|z| \leqslant r_0$, where r_0 is the smallest positive root of the equation (45).

A little computation shows that

$$\operatorname{Re}\left[1 + \frac{zf_0''(z)}{f_0'(z)}\right] = 0 \quad \text{for} \quad z = r_0, \delta_1 = \delta_2 = -1$$

where $f_0(z)$ is defined by (25). $f_0(z)$ fails to map $|z| \leqslant r$ onto a convex domain if $r > r_0$.

Taking $A = 1$, $B = -1$ in (45) we have the following:

Corollary — $r = 2 - \sqrt{3}$ which is the radius of convexity of C.

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