Indian J. pure appl. Math., **12**(5) : 659–663, May 1981

PLANE-SYMMETRIC STATIC CHARGED-DUST DISTRIBUTION IN SEN-DUNN THEORY

A. R. ROY AND BANI CHATTERJEE

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302

(*Received* 17 *January* 1980)

By assuming a relation between the 4-component of the metric tensor (i.e. g_{44}), the electro-magnetic-potential ψ and the scalar interaction function x^0, we have obtained an exact solution for plane-symmetric static charged dust distribution in the framework of Sen-Dunn theory of gravitation. It has been observed that for the solution obtained, the ratio of charge density to mass density is related to the scalar-interaction function x^0 such that for small values of x^0 the charge density far exceeds the mass-density.

INTRODUCTION

It is a well-known result in classical theory that under the action of gravitational attraction and electromagnetic repulsion, a static charged-dust distribution is held in equilibrium provided the charge-density is equal to mass-density ($\sigma = \pm \rho$). The analogous result in the framework of general relativity is also true as is evident from the work of Mazumder (1947), Das (1962), De and Raychaudhury (1968). The study of charged-dust distribution in Sen-Dunn (1971) theory of gravitation with a view to examine how the presence of the scalar interaction function x^0 alters the Mathematical and physical consequences of Einstein's theory is worth attempting. Our present study is an attempt in this direction. Recently Nayak (1975) has solved the problem for the B-D theory assuming a functional relationship between the metric potential g_{44} and the electrostatic potential ψ. In the present paper we have taken up the problem of solving the field equations of the Sen-Dunn theory corresponding to a plane-symmetric charged-dust distribution. As is well known, in this theory a new scalar-interaction function viz., the x^0 interaction is introduced to make the theory Machian. To solve the nonlinear equations corresponding to the problem we have assumed a relationship viz., $g_{44}(x^0_1)^2 = (8\pi G/\omega)\,\psi_1^2$ where g_{44} is the metric component of the static universe, x^0 the Sen-Dunn scalar field, ψ the source free electromagnetic field. This relationship has helped us to solve the equations for an exact solution for which the ratio of charge-density to matter-density has been found to be related to the scalar-interaction x^0 as given by $\sigma/\rho = k(x^0)^{-1}$, k being a constant.

FIELD EQUATIONS AND SOLUTIONS

We start by considering a static plane-symmetric metric

$$ds^2 = e^{2\nu}(dt^2 - dx^2) - e^{2q}(dy^2 + dz^2) \qquad \ldots(1)$$

where p and q are functions of x only. The field equations for the region of space-time containing charged-dust distribution in the framework of Sen and Dunn theory of gravitation are

$$R_{ij} - \tfrac{1}{2} R g_{ij} = 8\pi G (x^0)^{-2}\, T_{ij} + \omega (x^0)^{-2} \left(x^0_i\, x^0_j - \tfrac{1}{2} g_{ij}\, x^0_k\, x^{0k}\right) \qquad \text{...(2)}$$

$$F^{ij}_{\ ;j} = \sigma u^i \qquad \text{...(3)}$$

$$F_{[ij;k]} = 0 \qquad \text{...(4)}$$

where $\omega = 3/2$ and the energy-momentum tensor T_{ij} corresponding to the charged dust distribution is taken in the form

$$T_{ij} = \rho u_i u_j + \left(g^{ab} F_{ai}\, F_{bj} - \tfrac{1}{4}\, g_{ij}\, F^{ab}\, F_{ab}\right) \qquad \text{...(5)}$$

with

$$F_{ij} = \psi_{j,i} - \psi_{i,j} \qquad \text{...(6)}$$

as the components of the electromagnetic field tensor. Here u_i denotes the 4-velocity and ρ and σ respectively the mass and charge densities.

The static nature of the space-time enables us to write $u^\alpha = 0$ and $u^4 = (g_{44})^{-1/2}$. The Greek indices take the values 1, 2, 3, whereas the Latin ones range from 1, 2, 3, 4.

For $\psi_\alpha = 0$ and ψ_4 independent of x^4, we have only electrostatic field in the static universe. For convenience, we write $\psi_4 = \psi$, thus $F_{\alpha 4} = \psi_\alpha$ and $F_{\alpha\beta} = 0$.

For the metric (1), the field equations to be solved, finally reduce to the form

$$q_1^2 + 2p_1 q_1 = -\frac{8\pi G}{2(x^0)^2}\, e^{-2p} \psi_1^2 - \frac{\omega}{2} \left(\frac{x^0_1}{x^0}\right)^2 \qquad \text{...(7)}$$

$$p_{11} + q_{11} + q_1^2 = \frac{8\pi G}{2(x^0)^2}\, e^{-2p} \psi_1^2 + \frac{\omega}{2} \left(\frac{x^0_1}{x^0}\right)^2 \qquad \text{...(8)}$$

$$-2q_{11} - 3q_1^2 + 2p_1 q_1 = -\frac{8\pi G}{(x^0)^2}\, e^{2p} \rho + \frac{8\pi G}{2(x^0)^2}\, e^{-2p} \psi_1^2 - \frac{\omega}{2} \left(\frac{x^0_1}{x^0}\right)^2 \qquad \text{...(9)}$$

and

$$\psi_{11} + 2(q_1 - p_1)\, \psi_1 = \sigma e^{3p}. \qquad \text{...(10)}$$

From (7) and (8), we get

$$p_{11} + 2p_1 q_1 + q_{11} + 2q_1^2 = 0$$

which on integration yields

$$p_1 = -\, q_1 \qquad \text{...(11)}$$

assuming, for convenience, the constant of integration to be zero.

As the set of equations (7) – (10) is under-determinate to solve the unknowns p, q, ψ, x^0, σ, ρ, we first constrain the metric parameter q to be given by the linear form

$$q = Ax_1 + B \qquad \text{...(12)}$$

where A and B are arbitrary constants. Thus in view of (11), the other metric parameter p is obtained as

$$p = -Ax_1 + C \qquad \text{...(13)}$$

where C is an arbitrary constant.

Using (11) and (12) in (7) or (8), we get

$$\psi_1^2 = [2A^2(x^0)^2 - \omega(x_1^0)^2]\, \frac{e^{2p}}{8\pi G}. \qquad \text{...(14)}$$

In order to make the set of eqns. (7) to (10) a determinate one, we finally assume a functional relationship between the metric parameter g_{44}, the electromagnetic potential ψ and the scalar interaction function x^0 as given by

$$-g_{44} = \frac{8\pi G}{\omega}\left(\frac{\psi_1}{x_1^0}\right)^2. \qquad \text{...(15)}$$

Here it may be noted that in the framework of general theory of relativity it has been established by Mazumder (1947) and Papapetrou (1947) independently, that there always exists a functional relationship between the metric potential g_{44} and the electromagnetic potential ψ as given by

$$g_{44} = -(4\pi\psi^2 + A\psi + B).$$

Keeping this in view, in the framework of Sen-Dunn theory of gravitation, we naturally expect that the 4-component of the metric-tensor should be related to ψ and the scalar interaction function x^0. Now, eqns. (14) and (15), give

$$\left(\frac{x_1^0}{x^0}\right)^2 = \frac{A^2}{\omega}$$

which on integration yields

$$\log (x^0) = \frac{A}{\sqrt{\omega}}\, x_1 + D.$$

Hence, the scalar-interaction function x^0 is given by

$$x^0 = \exp\left(\frac{A}{\sqrt{\omega}}\, x_1 + D\right). \qquad \text{...(16)}$$

Substituting the value of x^0 in (14), we obtain

$$\psi = \left(\frac{\omega}{8\pi G}\right)^{1/2}\left(\frac{1}{1-\sqrt{\omega}}\right)\exp\left(A\left(\frac{1}{\sqrt{\omega}} - 1\right)x_1 + C + D\right). \qquad \text{...(17)}$$

Equation (9) gives

$$\rho = \frac{5A^2}{8\pi G} \exp\left(2\left(1 + \frac{1}{\sqrt{\omega}}\right) Ax_1 + 2D - 2C\right) \quad \ldots(18)$$

and eqn. (10) fixes the charge density as given by

$$\sigma = \frac{A^2}{\sqrt{8\pi G}} \left(\frac{1}{\sqrt{\omega}} + 3\right) \exp\left(D - 2C + \left(2 + \frac{1}{\sqrt{\omega}}\right) Ax_1\right). \quad \ldots(19)$$

The ratio of charge-density to mass-density is given by

$$\frac{\sigma}{\rho} = \left(\frac{8\pi G}{5}\right)^{1/2} \left(3 + \frac{1}{\sqrt{\omega}}\right) (x^0)^{-1} \quad \ldots(20)$$

i.e. $$\frac{\sigma}{\rho} = k(x^0)^{-1}$$

where

$$k = \left(\frac{8\pi G}{5}\right)^{1/2} \left(3 + \frac{1}{\sqrt{\omega}}\right).$$

Thus the ratio of charge-density to mass-density is related to the scalar interaction function x^0.

Motion of Test Particles

In Sen-Dunn theory, the motion of a test particle is governed by the equations

$$x^0 \ddot{x}^\mu + x^0 \left\{ {\mu \atop \alpha\beta} \right\} \dot{x}^\alpha \dot{x}^\beta + \tfrac{1}{2} \dot{x}^0_{\alpha} \dot{x}^\mu \dot{x}^\alpha + \tfrac{1}{2} g_{\alpha\beta} g^{\mu\nu} x^0_{\nu} \dot{x}^\alpha \dot{x}^\beta = 0. \quad \ldots(21)$$

Taking the initial motion in the x-direction i.e. $dy/ds = 0$, $dz/ds = 0$, eqn. (21) implies that the acceleration in the y and z direction are zero i.e. the motion of the particle continues to be radial and is given by

$$\frac{d^2x}{ds^2} - 4A \left(\frac{dx}{ds}\right)^2 + \frac{A}{2} \left(\frac{dt}{ds}\right)^2 = 0 \quad \ldots(22)$$

and

$$\frac{d^2t}{ds^2} - \frac{5A}{2} \frac{dx}{ds} \cdot \frac{dt}{ds} = 0. \quad \ldots(23)$$

From (1), we have

$$\left(\frac{dx}{ds}\right)^2 - \left(\frac{dt}{ds}\right)^2 = e^{2Ax-2C}. \quad \ldots(24)$$

Now assuming $x = f(t)$, we get from (22)–(24)

$$s = \int [(\dot{f}^2 - 1)^{1/2} e^{-Af+C}] \, dt + B \quad \ldots(25)$$

where a dot represents differentiation w.r.t. t, and f is given by the differential equation

$$\ddot{f} - \frac{3A}{2}\dot{f}^2 + \frac{A}{2} = 0. \qquad \ldots(26)$$

The equation (26) has a solution given by

$$f(t) = \frac{2}{3A}\log\left[\sec\left(\frac{\sqrt{3}\,At}{2} + D\right) + \tan\left(\frac{\sqrt{3}\,At}{2} + D\right)\right] + E.$$

Conclusion

For the plane-symmetric charged-dust distribution in Sen-Dunn theory, we find for our solution, that the ratio σ/ρ is related to the scalar-interaction function x^0, unlike that in general relativity. Clearly for small values of x^0 the charge-density will far exceed the mass-density. At spatial infinity the scalar interaction function x^0 and the electromagnetic potential ψ become singular. The first-curvature invariant $R_{ij}\,R^{ij}$ as well as the Kretchmann curvature invariant $R_{hijk}\,R^{hijk}$ show a similar limiting behaviour, viz., as $x \to 0$ $R_{ij}\,R^{ij}$ and $R_{hijk}\,R^{hijk}$ tend to a finite value, while as $x \to \infty$ both tend to $\to \infty$.

Acknowledgement

The authors are thankful to Professor J. R. Rao for his interest and constant encouragement in the work.

References

Das, A. (1962). A class of exact solutions of certain classical field equations in general relativity. *Proc. R. Soc.*, A **267**, 1.

De, U. K., and Raychaudhuri, A. K. (1968). Static distribution of charged dust in general relativity. *Proc. R. Soc.*, A **303**, 97-101.

Mazumder, S.D. (1947). A class of exact solutions in Einstein's field equations. *Phy. Rev.*, **72**, 390-98.

Nayak, B. K. (1975) Static charged-dust distribution in Brans-Dicke theory. *Aust. J. Phys.*, **28**, 585.

Papapetrou, A. (1947). A static solution of the equations of the gravitational field for an arbitrary charge distribution. *Proc. R. Irish Acad.*, A **51**, 191.

Sen, D. K., and Dunn, K. A. (1971). A scalar-tensor theory of gravitation in a modified Riemannian manifold. *J. Math. Phys.*, **13**, 557.