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# TORSIONAL VIBRATIONS OF SHALLOW SPHERICAL SANDWICH SHELLS

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Free torsional vibrations of shallow spherical sandwich shells are considered by taking into account the effects of transverse shear and rotatory inertia for the core as well as for the facings. The equations of motion derived by Hamilton's energy principle are solved in terms of Bessel functions with the help of auxiliary variables. Frequencies are computed for the first four normal modes of vibrations for a shell closed at the apex and clamped at the circumferential edge.

## INTRODUCTION

The work available on vibrations of spherical sandwich shells is as follows. Tasi (1964) has considered the effect of mass loss on the transient response of a shallow shell. Suvernev (1965) has considered natural vibrations of shells with freely supported and fixed edges. Wilkinson (1966) has analysed the problem of closed shells. Koplik and Yu (1967a, b) have considered axisymmetric torsionless and torsional vibrations of spherical caps. Forced vibrations are considered by Culkowski and Reismann (1971). Free vibrations of shallow and deep shells are considered by Mirza and others (1972, 1974). In almost all the papers cited above, the facings of the sandwich constraction are assumed as membranes and therefore their thicknesses are restricted to very small values. Restrictions are also imposed on the material of the core.

In the present paper, free torsional vibrations of shallow spherical sandwich shells are considered by taking less restrictive assumptions. Effects of transverse shear deformation and rotatory inertia are retained for the core as well as for the facings. This makes the theory applicable even to cases where the facings have appreciable thickness compared to core. Besides that, no restriction is imposed on the materials of the core and the facings. The equations of motion are derived by Hamilton's energy principle They are solved in terms of Bessel's functions with the aid of auxiliary variables. Frequencies are computed for first four normal modes of vibrations for a shell closed at the apex and clamped at the circumferential edge and compared with those of monocoque shells.

## DISPLACEMENTS, STRAINS AND STRESSES

We consider a sandwich spherical shell of thickness $2h$. The thickness of the core and each of the two facings are taken $2h_1$ and $h_2$, so that $h = h_1 + h_2$. Taking the radius of the middle surface of the shell as $R$ and the centre of the shell as origin, the shell is referred to spherical coordinates $(R + r, \theta, \phi)$, where $r$ denotes the radial distance from the middle surface taken positive along the outward normal, and $\theta$ and $\phi$ denote the angular coordinates in the meridional and circumferential directions respectively. The shell is taken extending from $\theta_1$ to $\theta_0$ in the meridional direction and from 0 to $2\pi$ in the circumferential direction. The inner and outer interfaces between the core and the facings lie on the surfaces of radii $R - h_1$ and $R + h_1$ respectively. Both the facings are taken of the same material but in general different from that of the core. The various quantities for the core, the inner facing and the outer facing will be distinguished by the subscripts 1, 2, and 3 respectively.

Since we are considering torsional axisymmetric vibrations of the shell, there will be displacements only in the direction of $\phi$ and independent of $\phi$. Therefore the nonzero displacements are approximated by

$$\left.\begin{aligned} v_1(r, \theta, t) &= v(\theta, t) + r\gamma_1(\theta, t) \\ \begin{matrix} v_2 \\ v_3 \end{matrix}(r, \theta, t) &= v(\theta, t) \mp h_1\gamma_1(\theta, t) + (r \pm h_1)\,\gamma_2(\theta, t) \end{aligned}\right\} \quad \ldots(1)$$

where $v$ is the displacement of the middle surface, $\gamma_1$ and $\gamma_2$ are the rotations of the normal to the middle surface for the core and the facings and $t$ is the time.

Since the shell considered here is shallow, we restrict ourselves to a large $R$ and small values of $\theta$. Therefore taking

$$\cos\theta \approx 1, \quad \sin\theta \approx \theta \quad \ldots(2)$$

and using the expressions given by Love (1944), the nonzero strain components are obtained to be

$$\left.\begin{aligned} \epsilon_{\theta\phi 1} &= [v_{,\theta} + r\gamma_{1,\theta} - (v + r\gamma_1)/\theta]/(r + R) \\ \begin{matrix} \epsilon_{\theta\phi 2} \\ \epsilon_{\theta\phi 3} \end{matrix} &= [v_{,\theta} - v/\theta \mp h_1(\gamma_{1,\theta} - \gamma_1/\theta) + (r \pm h_1)(\gamma_{2,\theta} - \gamma_2/\theta)]/(r + R) \\ \epsilon_{\theta\phi 1} &= (R\gamma_1 - v)/(r + R) \\ \begin{matrix} \epsilon_{r\phi 2} \\ \epsilon_{r\phi 3} \end{matrix} &= [R\gamma_2 - v \pm h_1(\gamma_1 - \gamma_2)]/(r + R) \end{aligned}\right\} \quad \ldots(3)$$

where a comma followed by a suffix denotes partial differentiation with respect to that variable.

The stress-strain relations are taken to be

$$(\sigma_{\theta\phi i}, \sigma_{r\phi i}) = \mu_i(\epsilon_{\theta\phi i}, \epsilon_{r\phi i}), \quad \mu_i = E_i/2(1 + \nu_i) \quad \ldots(4)$$

where $i$ takes the values 1, 2 and 3, $E_i$ and $\nu_i$ denote Young's modulii and Poisson's ratios.

## Equations of Motion

Applying Hamilton's energy principle, the equations of motion obtained in terms of stress resultants are

$$\left.\begin{aligned} N_{\theta\phi,\theta} + (2/\theta)\, N_{\theta\phi} + Q_\phi &= K_1 v_{,tt} + K_2\gamma_{1,tt} + K_3\gamma_{2,tt} \\ P_{\theta\phi,\theta} + (2/\theta)\, P_{\theta\phi} - K_\phi &= K_2 v_{,tt} + K_4\gamma_{1,tt} + K_5\gamma_{2,tt} \\ M_{\theta\phi,\theta} + (2/\theta)\, M_{\theta\phi} - F_\phi &= K_3 v_{,tt} + K_5\gamma_{1,tt} + K_6\gamma_{2,tt} \end{aligned}\right\} \quad \ldots(5)$$

where

$$\left.\begin{aligned} N_{\theta\phi} &= N_{\theta\phi 1} + N_{\theta\phi 2} + N_{\theta\phi 3},\ Q_\phi = Q_{\phi 1} + Q_{\phi 2} + Q_{\phi 3} \\ P_{\theta\phi} &= M_{\theta\phi 1} - h_1(N_{\theta\phi 2} - N_{\theta\phi 3}),\ K_\phi = RQ_{\phi 1} + h_1(Q_{\phi 2} - Q_{\phi 3}) \\ M_{\theta\phi} &= M_{\theta\phi 2} + M_{\theta\phi 3} + h_1(N_{\theta\phi 2} - N_{\theta\phi 3}), \\ F_\phi &= (R - h_1)\, Q_{\phi 2} + (R + h_1)\, Q_{\phi 3} \end{aligned}\right\} \quad \ldots(6)$$

$$(N_{\theta\phi i}, Q_{\phi i}, M_{\theta\phi i}) = \int_{x_i}^{y_i} \sigma_{\theta\phi i}, k_s\sigma_{r\phi i}, r\sigma_{\theta\phi i}) \left(1 + \frac{r}{R}\right) dr \quad \ldots(7)$$

$$\left.\begin{aligned} K_1 &= [p_1(3R^2 + h_1^2) + p_2(3R^2 + k_2)]/3R \\ K_2 &= (2p_1h_1^2 + 3p_2k_1)/3 \\ K_3 &= p_2(2k_2 + 3h_1k_1)/3 \\ K_4 &= h_1^2\,[p_1(R^2 + 0\,.\,6h_1^2) + p_2(3R^2 + k_2)]/3R \\ K_5 &= p_2h_1(6h_2R^2 + 3k_3 - 4h_1k_2)/12R \\ K_6 &= p_2\,[10R^2(k_2 - 3h_1h) + 10h_1^2\,k_2 - 15h_1k_3 + 6k_4]/30R \\ p_1 &= 2\rho_1h_1,\ p_2 = 2\rho_2h_2,\ k_1 = 2h_1 + h_2 \\ k_2 &= 3h_1^2 + 3h_1h_2 + h_2^2 \\ k_3 &= 4h_1^3 + 6h_1^2\,h_2 + 4h_1h_2^2 + h_2^3 \\ k_4 &= 5h_1^3\,h + 5h_1h_2h^2 + h_2^4. \end{aligned}\right\} \quad \ldots(8)$$

The limits of integration $x_i$ to $y_i$, $i = 1, 2, 3$, stand for $-h_1$ to $h_1$, $-h$ to $-h_1$ and $h_1$ to $h$ respectively, $k_s$ is the shear constant and $\rho_1$ and $\rho_2$ are the densities of the core and the facings.

With the help of relations (3), (4) and (6) to (8), the equations of motion (5) are reduced to three equations coupled in displacement variables $v$, $\gamma_1$ and $\gamma_2$. To make them uncoupled in a simpler way, we take

$$(v, \gamma_1, \gamma_2) = (RV_{,x}, G_{1,x}, G_{2,x}) \exp\,(i\omega t) \quad \ldots(9)$$

where the auxiliary variables $V$, $G_1$ and $G_2$ are functions of $X = \theta/\theta_0$ only, $\exp(i\omega t)$ is the time factor for harmonic vibrations and $\omega$ is the circular frequency. Substituting (9) in the equations coupled in displacement variables and integrating them with respect to $X$ we get

$$\left.\begin{aligned} (a_0D^2 + a_1)\,V + a_2G_1 + a_3G_2 &= 0 \\ a_2V + (b_0D^2 + b_1)\,G_1 + (b_2D^2 + b_3)\,G_2 &= 0 \\ a_3V + (b_2D^2 + b_3)\,G_1 + (c_0D^2 + c_1)\,G_2 &= 0 \end{aligned}\right\} \quad \ldots(10)$$

where

$$\left.\begin{aligned} a_0 &= 3(1 + R_hR_u)/\theta_0^2,\ b_0 = R_r^2\,(1 + 3R_hR_u)/\theta_0^2, \\ b_2 &= 1.5R_h^2\,R_uR_r^2\,/\theta_0^2 \\ c_0 &= R_h^3\,R_r^2\,R_u/\theta_0^2,\ a_1 = a_0(2 - k_s) + R_1(3 + R_r^2 + a_4)\,\Omega^2 \\ a_2 &= 3k_s + R_1R_r^2\,[2 + 3R_pR_h(2 + R_h)]\,\Omega^2, \\ a_3 &= 3R_hR_uk_s + R_1R_pR_h^2\,R_r^2\,(3 + 2R_h)\,\Omega^2 \\ a_4 &= R_pR_h\,[3 + R_r^2\,(3 + 3R_h + R_h^2)],\ b_1 = R_r^2\,[2 + 3R_uR_h \\ &\qquad \times (2 - k_s) + R_1(1 + 0.6R_r^2 + a_4)\,\Omega^2] - 3k_s \\ b_3 &= R_hR_r^2\,[3R_u(R_h + k_s) + 0.25R_pR_h \\ &\qquad \times \{6 + R_r^2\,(6 + 8R_h + 3R_h^2)\}\,R_1\Omega^2] \\ c_1 &= R_hR_u\,[2R_r^2\,R_h^2 - 3(1 + R_r^2)\,k_s] + R_pR_h^3\,R_r^2\,R_1 \\ &\qquad \times [1 + R_r^2\,(1 + 1.5R_h + 0.6R_h^2)]\,\Omega^2 \\ R_h &= h_2/h_1,\ R_r = h_1/R,\ R_p = \rho_2/\rho_1,\ R_u = \mu_2/\mu_1,\ R_1 = \lambda_1/\mu_1 \\ \lambda_1 &= E_1/(1 - \nu_1^2),\ \Omega^2 = \rho_1R^2\omega^2/\lambda_1,\ D^2 \equiv \frac{d^2}{dx^2} + \frac{d}{xdx}. \end{aligned}\right\} \quad \ldots(11)$$

Here $\Omega$ is the non-dimensional frequency parameter. The arbitrary constants arising in eqns. (10), due to integration, are set equal to zero without any loss of generality.

Eliminating any two of the variables $V$, $G_1$ and $G_2$ from eqns. (10), we get a sixth order differential equation

$$(r_0D^6 + r_1D^4 + r_2D^2 + r_3)\,F = 0 \quad \ldots(12)$$

where $F$ denotes any one of the variables $V$, $G_1$ and $G_2$ and

$$\left.\begin{aligned}
&r_0 = a_0A_1,\ r_2 = a_0A_3 + a_1A_2 + a_2A_4 + a_3A_5,\ r_1 = a_0A_2 + a_1A_1\\
&r_3 = a_1A_3 + a_2A_6 + a_3A_7,\ A_1 = b_0c_0 - b_2^2,\\
&A_2 = b_0c_1 - c_0b_1 - 2b_2b_3\\
&A_3 = b_1c_1 - b_3^2,\ A_4 = a_3b_2 - c_0a_2,\ A_5 = a_2b_2 - b_0a_3\\
&A_6 = a_3b_3 - c_1a_2,\ A_7 = a_2b_3 - a_3b_1.
\end{aligned}\right\} \quad \ldots(13)$$

## SOLUTION

The solution of Bessel's equation

$$(D^2 + n^2)\, F = 0 \quad \ldots(14)$$

will also be a solution of eqn. (12), provided

$$r_0n^6 - r_1n^4 + r_2n^2 - r_3 = 0. \quad \ldots(15)$$

Equation (15) is a cubic in $n^2$. Let its three roots be $n_1^2$, $n_2^2$ and $n_3^2$. Then the general solution for auxiliary variables can be taken as

$$(V, G_1, G_2) = \sum_{\alpha=1}^{3} (f_\alpha, g_\alpha, 1)\,[B_\alpha J_0(n_\alpha X) + C_\alpha Y_0(n_\alpha X)] \quad \ldots(16)$$

where $B_\alpha$ and $C_\alpha$ are arbitrary constants, $J_0$ and $Y_0$ are Bessel functions of first and second kinds of order zero and $f_\alpha$ and $g_\alpha$ can be obtained from any two of the eqns. (10) after substituting solution (16) in them.

The substitution of (16) in (9) gives solution for $v$, $\gamma_1$ and $\gamma_2$. The arbitrary constants are to be determined by the edge conditions at the edges $\theta = \theta_1$ and $\theta = \theta_0$. If the shell is closed at the apex, the terms in $Y_0$ are to be discarded to avoid singularity at $\theta_1 = 0$.

## FREQUENCY EQUATION

If we take a shell closed at the apex and clamped at the edge $\theta = \theta_0$, the edge conditions are

$$v = \gamma_1 = \gamma_2 = 0 \quad \text{at} \quad X = 1. \quad \ldots(17)$$

Substituting solution (16) in (9) after taking $C_\alpha = 0$ and using the edge conditions (17), we get

$$\left.\begin{aligned}
&\sum_{\alpha=1}^{3} B_\alpha f_\alpha n_\alpha J_1(n_\alpha) = 0\\
&\sum_{\alpha=1}^{3} B_\alpha g_\alpha n_\alpha J_1(n_\alpha) = 0\\
&\sum_{\alpha=1}^{3} B_\alpha n_\alpha J_1(n_\alpha) = 0.
\end{aligned}\right\} \quad \ldots(18)$$

Eliminating arbitrary constants from the above equations, we get a frequency equation in the form of a third order determinant equated to zero, say

$$| D_{ij} | = 0, \quad (i, j = 1, 2, 3). \qquad \ldots(19)$$

The elements $D_{ij}$ involve the frequency parameter $\Omega$ implicitly. The frequencies for successive normal modes of vibrations are those values of $\Omega$ for which (19) is satisfied.

The frequency equation for a monocoque shell can be obtained by putting $R_h = 0$ in the equations of motion and then proceeding in a similar manner as above. It will give rise to a second order determinant equated to zero.

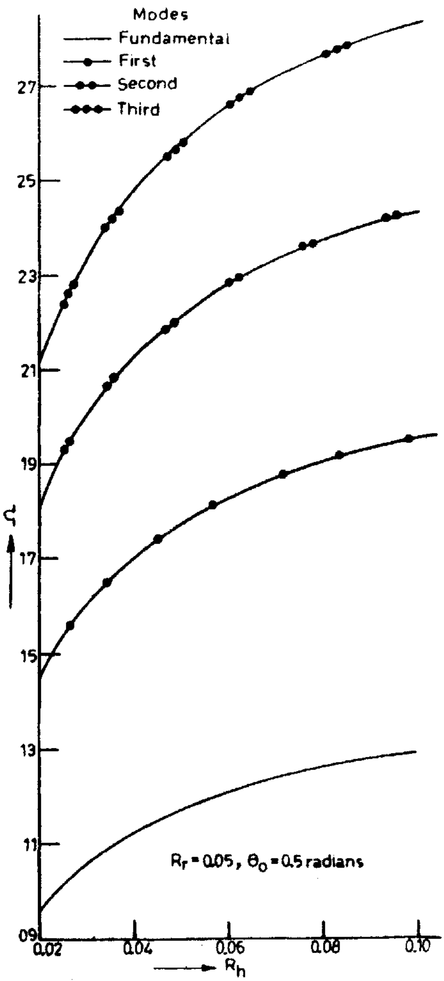


FIG. 1. $\Omega$ vs $R_h$ for sandwich shells.

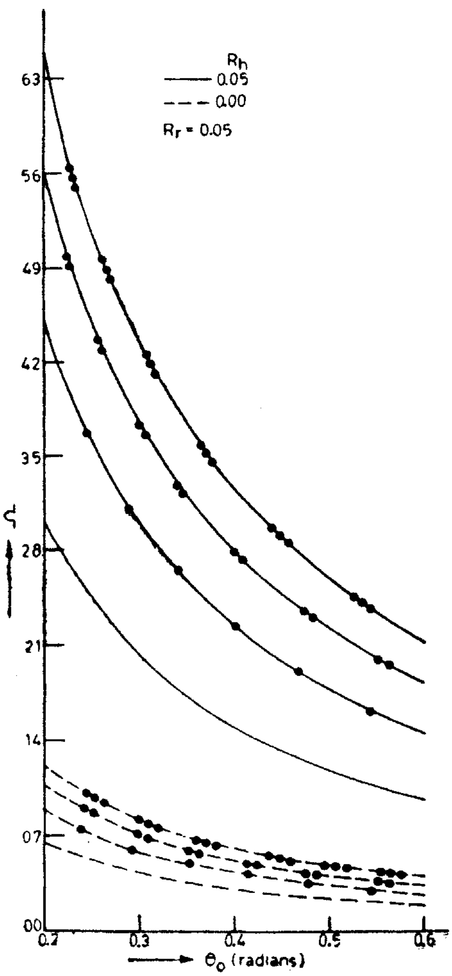


FIG. 2. $\Omega$ vs $\theta_0$ for sandwich and monocoque shells.

## Numerical Results

In the present investigation, $k_s$ is taken equal to $\pi^2/12$ and the materials for the core and the facings are taken Cellular Cellulose Acetate and Aluminium, for which the values of various constants are :

$$R_p = 34.4, \quad R_u = 1683.0, \quad R_1 = 2.20. \qquad \ldots(20)$$

The behaviour of $\Omega$ with $R_h$ for first four modes of vibrations is shown in Fig. 1. It is seen that the value of $\Omega$ increases with the increase in $R_h$, first rapidly and then slowly, in all the modes of vibrations. The difference between consecutive modes decreases as we go to higher modes.

Figure 2 shows the behaviour of $\Omega$ with $\theta_0$ for first four modes of vibrations of sandwich and monocoque shells. It is clear from the graphs that the value of $\Omega$ decreases with the increase in $\theta_0$, first rapidly and then slowly, in all the modes of vibrations of sandwich as well as monocoque shells. But the value of $\Omega$ as well as the rate of variation of $\Omega$ is much more higher for sandwich shells as compared to monocoque shells.

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