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WONG'S FIXED POINT THEOREMS

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Conditions are sought in order that a pair of operators defined on a sequentially complete Hausdorff uniform space has a fixed point. The results obtained herein generalize some of the recent results of Wong (1973).

1. INTRODUCTION

Let (X, U) be a sequentially complete Hausdorff uniform space defined by a family D of pseudometrics d_i, $i \in \Lambda$ on X (cf. Kelley 1955, Chap. 6). The following definitions will be useful in our subsequent discussions :

$$V_{(d,r)} = \{(x, y) : x, y \in X, d(x, y) < r, \quad r > 0\}$$

$$G = \{ V : V = \bigcap_{i \in F} V_{(d_i, r_i)}, d_i \in D, r_i > 0, F \subset \Lambda \text{ and } F \text{ is finite}\}$$

For $\quad V = \bigcap_{i \in F} V_{(d_i, r_i)} \in G, \quad$ let

$$aV = \bigcap_{i \in F} V_{(d_i, ar_i)}, \; a > 0,$$

$$= \Delta \text{ (the diagonal)}, \quad a = 0$$

Here a particular V may have different representations $\bigcap_{i \in F} V(d_i, r_i)$ and we may accordingly get different sets $aV \in G$ corresponding to different representations of V. This is illustrated by an example given subsequently. In what follows, having taken a particular representation for a specific V, the same representation is adhered to in the sequel.

We use the following simple Lemmas (cf. Acharya 1974):

Lemma 1.1 — Let $V \in G$ and $a, b > 0$, then

(i) $a(bV) = (ab) V$,

(ii) $aV \circ bV \subset (a + b) V$,

(iii) $aV \subset bV$ for $a < b$.

Lemma 1.2 — Let d be a pseudometric on X and $a, b > 0$. If

$$(x, y) \in aV_{(d,r_1)} \circ bV_{(d,r_2)}$$

then

$$d(x, y) < ar_1 + br_2.$$

Lemma 1.3 — Let $V \in G$ be arbitrary, then there is a pseudometric p on X such that

$$V = V_{(p,1)}.$$

This p is called a Minkowski's pseudometric of V.

Example : Let $0 < a < 1$. Let R be the real line. Let us take a pseudometric d_1 on R such that $d_1(x, y) \leqslant 1$ for all $x, y \in R$, e.g.

$$d_1(x, y) = \frac{|x - y|}{1 + |x - y|}$$

will serve the purpose. Now define

$$d_2(x, y) = \min \{a, d_1\}.$$

Then we have,

$$V_{(d_1,1)} = V_{(d_2,1)} = R \times R.$$

Now, for $a < b < 1$

$$bV = V_{(d_1,b)} \neq R \times R$$

$$bV = V_{(d_2,b)} = R \times R.$$

This shows that the representation of aV need not be unique. The following theorem has been proved by Mishra (1978).

Theorem A — Let T_1 and T_2 be two operators on X such that for

$$V_i \in G (i = 1, \ldots, 5) \quad \text{and} \quad x, y, \in X,$$

(a) $(T_1xT_2y) \in a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5$,

if $(x, T_1x) \in V_1$, $(y, T_2y) \in V_2$, $(x, T_2y) \in V_3$, $(y, T_1x) \in V_4$ and $(x, y) \in V_5$ where $a_i \geqslant 0$ $(i = 1, \ldots, 5)$, $\sum_{i=1}^{5} a_i < 1$,

(b) $a_1 = a_2$ or $a_3 = a_4$.

Then T_1 and T_2 have a unique common fixed point.

This theorem generalizes a well known result of Wong (1973) to uniform spaces. The purpose of the present note is to study the related results.

2. Fixed Points of Mappings

Theorem 2.1 — Let T_1 and T_2 be two operators on X. Suppose that there exist functions $a_i : X \times X \to [0, 1)$, $(i = 1, \ldots, 5)$ such that for $V_1 \in G$ $(i = 1, \ldots, 5)$ and $x, y \in X$, $x \neq y$,

$$(T_1x, T_2Y) \in a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5 \qquad \ldots(2.1)$$

if $(x, T_1x) \in V_1$, $(y, T_2y) \in V_2$, $(x, T_2y) \in V_3$, $(y, T_1x) \in V_4$ and $(x, y) \in V_5$ where $a_i = a_i(x, y)$ $(i = 1, \ldots 5)$,

$$a = \operatorname*{Sup}_{x, y \in X} \left\{ \sum_{i=1}^{5} a_i(x, y) \right\} < 1, \qquad \ldots(2.2)$$

$$a_1(x, y) = a_2(x, y) \quad \text{and} \quad a_3(x, y) = a_4(x, y). \qquad \ldots(2.3)$$

Then T_1 or T_2 has a fixed point. If both T_1 and T_2 have fixed point, then each of T_1, T_2 has a unique fixed point and these two fixed points coincide.

Proof : Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows:

$$X_{2n+1} = T_1(x_{2n}), \quad x_{2n+2} = T_2(x_{2n+1}), \quad n = 0, 1, 2, \ldots$$

We shall prove that T_1 or T_2 has a fixed point.

Let $V \in G$ be arbitrary. Denote by p a Minkowski's pseudometric of V so that $V = V_{(p,1)}$. Let $x, y \in X$, $x \neq y$. Let us put $p(x, T_1x) = s_1$, $p(y, T_2y) = s_2$, $p(x, T_2y) = s_3$, $p(y, T_1x) = s_4$ and $p(x, y) = s_5$. Take $\epsilon > 0$. Then

$$(x, T_1x) \in V_{(p,s_1+\epsilon)}, \ (y, T_2y) \in V_{(p,s_2+\epsilon)}, \ (x, T_2y) \in V_{(p,s_3+\epsilon)},$$
$$(y, T_1x) \in V_{(p,s_4+\epsilon)} \quad \text{and} \quad (x, y) \in V_{(p,s_5+\epsilon)}.$$

Now, using condition (2.1) and Lemma 1.2, we get

$$p(T_1x, T_2y) < a_1(x, y)\, p(x, T_1x) + a_2(x, y)\, p(y, T_2y) + a_3(x, y)\, p(x, T_2y) + a_4(x, y)\, p(y, T_1x) + a_5(x, y) + \left(\sum_{i=1}^{5} a_i \right) \epsilon$$

Since $\epsilon > 0$ is arbitrary, we get

$$p(T_1x, T_2y) \leqslant a_1(x, y)\, p(x, T_1x) + a_2(x, y)\, p(y, T_2y) + a_3(x, y)\, p(x, T_2y) + a_4(x, y)\, p(y, T_1x) + a_5(x, y)\, p(x, y). \qquad \ldots(2.4)$$

We have

$$p(x_1, x_2) = p(T_1x_0, T_2x_1).$$

Using condition (2.4), we get

$$p(x_1, x_2) \leqslant \frac{a_1(x_0, x_1) + a_3(x_0, x_1) + a_5(x_0, x_1)}{1 - a_1(x_0, x_1) - a_3(x_0, x_1)}.$$

Since $a_2 + a_3 < a/2 < 1/2$, we have

$$\frac{a_1(x_0, x_1) + a_3(x_0, x_1) + a_5(x_0, x_1)}{1 - a_2(x_0, x_1) - a_3(x_0, x_1)}$$

$$\leqslant \frac{a - a_2(x_0, x_1) - a_3(x_0, x_1)}{1 - a_2(x_0, x_1) - a_3(x_0, x_1)}$$

$$\leqslant \max\left\{\frac{a - x}{1 - x} : x \in [0, \tfrac{1}{2})\right\}$$

$$\leqslant a.$$

Therefore,

$$p(x_1, x_2) \leqslant ap(x_0, x_1).$$

Similarly,

$$p(x_2, x_3) \leqslant ap(x_1, x_2) \leqslant a^2 p(x_0, x_1).$$

Thus in general, for $x_n \neq x_{n+1}$

$$p(x_n, x_{n+1}) \leqslant a^n p(x_0, x_1).$$

Now, for $m, n \geqslant N$

$$p(x_n, x_m) \leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{m-1}, x_m).$$

$$\leqslant (a^n + a^{n+1} + \dots + a^{m-1})\, p(x_0, x_1).$$

$$\leqslant \frac{a^n}{1 - a}\, p(x_0, x_1).$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $(x_n, x_m) \in V$ and hence $\{x_n\}$ is a Cauchy sequence. Since X is sequentially complete, $x_n \to y$ for some y in x. Also, since $x_n \neq x_{n+1}$ for each n, either $x_{2n+1} \neq y$ for infinitely many n or $x_{2n} \neq y$ for infinitely many n. Assume that $x_{2n+1} \neq y$ for infinitely many n. Thus there is a subsequence $k(n)$ of n such that $x_{2k(n)+1} \neq y$ for each n. Now, let $V \in G$ be arbitrary and denote by p a Minkowski's pseudometric of V. Then, for any positive integer, n, we have

$$p(y\ T_2 y) \leqslant p(y, x_{2k(n)+1}) + p(T_1 x_{2k(n)}, T_2 y).$$

Using condition (2.4) with $a_1 = a_i(x_{2k(n)}, y)$ we get

$$p(y, T_2 y) \leqslant (1 + a_4)\, p(x_{2k(n)+1}, y) + a_2 p(x_{2k(n)+1}, x_{2k(n)})$$
$$+ a_2 p(y, T_2 y) + a_3 p(x_{2k(n)}, T_2 y) + a_5 p(x_{2k(n)}, y).$$

Making $n \to \infty$, we get

$$p(y, T_2 y) \leqslant (a_2 + a_3) \,.\, p(y, T_2 y).$$

Since $a_2 + a_3 < 1$, it follows that $(y, T_2y) \in V$. Since V is arbitrary and X is a Hausdorff space, $T_2y = y$.

Let $Z \in X$ be such that $T_1Z = Z$ and $T_2Y = Y$. Take $V \in G$ be arbitrary. Then

$$(Z, T_1Z) = (Z, Z) \in V,$$
$$(Y, T_2Y) = (Y, Y) \in V$$

and

$$(Y, z) \in a_1V \circ a_2V \circ a_3V \circ a_4V \circ a_5V \subset (\sum_{i=1}^{5} a_i) V$$

where $a_i = a_i(y, z)$, $(i = 1, \ldots, 5)$, This follows from Lemma 1.1. Since V is arbitrary, we have $y = z$.

This completes the proof.

Theorem 2.2 — Let T be an operator on X. Suppose that there exist symmetric functions $a_i : X \times X \to [0, 1)$, $(i = 1, \ldots 5)$ such that for $V_i \in G(i = 1, \ldots, 5)$ and $x, y \in X$,

$$(Tx, Ty) \in a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5 \qquad \ldots(2.5)$$

if $(x, Tx) \in V_1$, $(y, Ty) \in V_2$, $(x, Ty) \in V_3$, $(y, Tx) \in V_4$ and $(x, y) \in V_5$ where $a_i = a_i(x, y)$ $(i = 1, \ldots, 5)$,

$$a = \operatorname*{Sup}_{x, y \in X} \{ \sum_{i=1}^{5} a_i(x, y) \} < 1. \qquad \ldots(2.6)$$

Then T has a unique fixed point.

PROOF : Let $x_0 \in X$ be arbitrary. Define a sequence $\{X_n\}$ in X as follows.

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2 \ldots$$

Let $V \in G$ be arbitrary, Denote by p a Minkowski's pseudometric of V. Then it can be easily shown that

$$p(Tx, Ty) \leqslant a_1(x, y)\, p(x, Tx) + a_2(x, y)\, p(Y, Ty) + a_3(x, y)\, p(x, Ty)$$
$$+ a_4(x, y)\, p(y, Tx) + a_5(x, y)\, p(x, y). \qquad \ldots(2.7)$$

Also, by symmetry, we have

$$p(Ty, Tx) \leqslant a_1(y, x)\, p(y, Ty) + a_2(y, x)\, p(x, Tx) + a_3(y, x)\, p(y, Tx)$$
$$+ a_4(y, x)\, p(x, Ty) + a_5(y, x)\, p(y, x). \qquad \ldots(2.8)$$

From (2.7) and (2.8), we get

$$p(Tx, Ty) \leqslant b_1p(x, Tx) + b_2p(y, Ty) + b_3p(x, Ty) + b_4p(y, Tx) + b_5p(x, y)$$

where

$$b_1 = \tfrac{1}{2}[a_1(x, y) + a_2(y, x)], \quad b_2 = \tfrac{1}{2}[a_1(y, x) + a_2(x, y)],$$
$$b_3 = \tfrac{1}{2}[a_3(x, y) + a_4(y, x)], \quad b_4 = \tfrac{1}{2}[a_3(y, x) + a_4(x, y)],$$
$$b_5 = \tfrac{1}{2}[a_5(x, y) + a_5(y, x)].$$

Since each $a_i (i = 1, \ldots, 5)$ is symmetric, $b_1 = b_2$, $b_3 = b_4$

and $$\sum_{i=1}^{5} b_i = \sum_{i=1}^{5} a_i(x, y) \leqslant a.$$

Therefore, we may assume that $b_1 = b_2$ and $b_3 = b_4$. By Theorem 2.1, T has a unique fixed point.

This completes the proof.

REFERENCES

Acharya, S. P. (1974). Some results on fixed points in uniform spaces. *Yokohama Math. J.*, **22**, 105–16.

Kelley, J. L. (1955). General Topology. D. Von Nostrand Co., New York.

Mishra, S. N. (1978). On common fixed points in uniform spaces II. *Indian J. pure appl. Math.*, **9**, 26–31.

Wong, C. S. (1973). Common fixed points of two mappings. *Pacific J. Math.*, **48**, 299–312.