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A NOTE ON COMPLETENESS OF BOUNDED LATTICES POSTULATED IN SOME AXIOMATICS OF THE MATHEMATICAL FOUNDATIONS OF QUANTUM THEORY

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In an axiomatic study of quantum theory Jauch postulated the completeness of the lattice underlying a quantum logic. The theory of Baer semigroup is utilized to specify quite generally the completeness of the lattice.

1. INTRODUCTION

To provide a mathematical foundation to quantum theory Jauch (1968) introduces an axiom which assumes the completeness of the lattice underlying a quantum logic characterized in Sharma and Mukherjee (1977). In this paper we prove a theorem which specifies quite generally the completeness of the lattice in terms of a Baer semigroup having a special connection with the lattice concerned.

Necessary notations and definitions are presented in detail in section 2.

2. PRELIMINARIES

Definition 2.1 — A relation r on a set A is a subset of the Cartesian product $A \times A$; notation xry means $(x, y) \in r$.

Definition 2.2 — A relation r on a set A is said to be

(a) anti-symmetric : if $x, y \in A$, xry and yrx, then $x = y$;

(b) reflexive : if $x \in A$, then xrx;

(c) transitive : if $x, y, z \in A$, xry and yrz, then xrz.

Definition 2.3 — A poset is a pair $(A, \leqslant)$, where A is a set and $\leqslant$ is an anti-symmetric, reflexive and transitive relation (a partial ordering) on A.

Definition 2.4 — Let $(A, \leqslant)$ be a poset and $B \subset A$.

(a) $x \in A$ is an upper bound for B provided : if $y \in B$, then $y \leqslant x$.

(b) $x \in A$ is a least upper bound for B provided:

(i) x is an upper bound for B;

(ii) if z is an upper bound for B, then $x \leqslant z$;

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(c) the least upper bound of B, if it exists, is denoted by $\vee B$; in case $B = \{b_1, b_2\}$, $\vee B$ is denoted by $b_1 \vee b_2$;

(d) lower bound, greatest lower bound $\wedge B$ and $b_1 \wedge b_2$ are defined dually;

(e) $(A, \leqslant)$ is a lattice if $a_1, a_2 \in A$ implies $a_1 \vee a_2$ and $a_1 \wedge a_2$ exist.

Definition 2.5 — Let $(A, \leqslant)$ be a lattice and $B \subset A$. $(A, \leqslant)$ is said to be complete if $\vee B$ and $\wedge B$ exist in A.

Definition 2.6 — A lattice $(A, \leqslant)$ is said to have a universal upper bound (universal lower bound) if $\vee A$ $(\wedge A)$ exists in A.

Definition 2.7 — A semigroup $(S, .)$ is a set with a mapping

$$. : S \times S \to S((x, y) \in S \times S \to x \,.\, y \in S)$$

such that if $x, y, z \in S$, then $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)$, i.e., . is associative.

Definition 2.8 — Let $(S, .)$ be a semigroup;

(a) an element $0 \in S$ is a zero provided $0 \,.\, x = x \,.\, 0 = 0$ for all $x \in S$;

(b) an element $e \in S$ is a right identity provided $x \,.\, e = x$ for all $x \in S$;

(c) an element $f \in S$ is a left identity provided $f.\, x = x$ for all $x \in S$;

(d) an element $i \in S$ is a two sided identity if it is both a right and a left identity;

(e) an element $p \in S$ is an idempotent of S provided $p = p \,.\, p = p^2$.

Throughout this paper a semigroup will mean a multiplicative semigroup with 0 and $\leqslant$ will denote the set inclusion. For any subset A of a semigroup S, $R(A) = \{y : y \in S, ay = 0$ for every $a \in A\}$ is the right annihilator of A and $L(A) = \{y : y \in S, ya = 0$ for every $a \in A\}$ is the left annihilator of A. If A is a singleton $\{x\}$ we shall write $R(x)$, instead of $R(\{x\})$, for the right annihilator of $x \in S$ and $L(x)$, instead of, $L(\{x\})$, for the left annihilator of $x \in S$. It is obvious that

$$R(A) = \bigcap_{a \in A} R(a) \text{ and } L(a) = \bigcap_{a \in A} L(a).$$

For each $x \in S$ and each $A \subset S$, let

$$xA = \{xa : a \in A\}, Ax = \{ax : a \in A\}.$$

Definition 2.9 — For each $x \in S$, xS is defined to be the principal right ideal generated by x and Sx is defined to be the principal left ideal generated by x.

Definition 2.10 — A semigroup S is said to be a Baer semigroup, if the right annihilator and the left annihilator of each element of S are respectively the principal right ideals and the principal left ideals generated by idempotents of S.

Janowitz (1965) has shown that with set theoretic inclusion as a partial ordering the set of right annihilators and the set of left annihilators of elements of a Baer semigroup from dual isomorphic lattices with universal bounds. It has also been proved (Janowitz 1966) that for every bounded lattice L there is a Baer semigroup S with the property that the lattice of the right annihilators of elements of S is isomorphic to L. In this case S is said to coordinatize L.

3. The Theorem

Definition 3.1 — We shall define a Baer semigroup S to be complete if, for every nonempty subset A of S, $R(A)$ is a principal right ideal generated by an idempotent of S.

Example — Let X be a given set and $P(X)$ be the collection of all subsets of X. If we define the multiplication in $P(X)$ by $a \,.\, b = a \cap b$, where $a, b \in P(X)$, then the system $(P(X), .)$ forms a semigroup. The empty set ϕ serves as the zero element of the semigroup. The right annihilator of any element a of $P(X)$ is the right ideal generated by the set theoretic complement of a in X which is, indeed, an idempotent of $P(X)$. Note that any right annihilator in this case is also a left annihilator and the corresponding statement is true for any right ideal generated by idempotent. The annihilator (right or left) of any subset of $P(X)$ is the ideal generated by an idempotent which is the complement in X of the union of the elements of the subset concerned. Hence $(P(X), .)$ forms a complete Baer semigroup.

Lemma 3.1 — A Baer semigroup S has a two sided identity which is unique.

Proof : The proof is trivial.

The unique two sided identity of S will be denoted by i.

Lemma 3.2 — Let R be the set of right annihilators of elements of a Baer semigroup S. Then, for every nonempty subset A of S, $R(A) \in R$, if S is complete.

Proof : Since the Baer semigroup S is complete, there is an idempotent e of S such that $R(A) = eS$. Let $z \in eS$.

Then $z = ea$ for some $a \in S$. For every $y \in L(e)$, $yz = yea = 0$.

That is, $z \in R(L(e))$. So $eS \leqslant R(L(e))$.

Since S has a two sided identity (Lemma 3.1.), $e \in eS$. So $e \in R(a)$ for each $a \in A$. Then for each a in A, $ae = 0$. That is each element of A is also an element of $L(e)$. So $A \leqslant L(e)$. Now $R(L(e)) = \bigcap_{z \in L(e)} R(z) \leqslant R(a)$ for every a in A. Consequently, $R(L(e)) \leqslant eS$.

Therefore, $R(L(e)) = eS$.

Now let $L(e) = Sf$ for some idempotent f of S. So $R(L(e)) = R(Sf)$. Let $x \in R(Sf)$. Then $afx = 0$ for every $a \in S$. Taking $a = i$, the identity of S, $fx = 0$. That is $x \in R(f)$. So $R(Sf) \leqslant R(f)$.

Let $x \in R(f)$. Then $fx = 0$. That is $afx = 0$ for every $a \in S$. So $x \in R(Sf)$. Consequently, $R(f) \leqslant R(Sf)$.

So $R(f) = R(Sf) = R(L(e)) = eS$. Therefore, $eS \in R$.

Since, for every bounded lattice L, there is a Baer semigroup S coordinatizing L, it will be sufficient to prove that lattice of the right annihilators of elements of a Baer semigroup S is complete, if S is complete.

Theorem — If a Baer semigroup S is complete then the lattice R of the right annihilators of elements of S is a complete lattice.

PROOF : Let P be any nonempty subset of R. Then $M = \{x : R(x) \in P\}$ is a nonempty subset of S. Since the Baer semigroup S is complete, there is an idempotent e of S such that $R(M) = eS$. eS is an infimum of P by the definition of $R(M)$. It follows from Lemma 3.2 that $eS \in R$. Therefore P has an infimum in R.

Let D be set of all upper bounds of P in R. R is a bounded lattice (Janowitz 1965). Let I be the universal upper bound of R. Then clearly $I \in D$. It now follows, from an argument similar to that used above, that the nonempty subset D of R has an infimum in R, which is the supremum of P in R. Therefore, the lattice R is a complete lattice.

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