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PERIODIC SOLUTION FOR CERTAIN NONLINEAR SECOND-ORDER AUTONOMOUS DIFFERENTIAL EQUATION

B. MEHRI

Department of Mathematics and Computer Science, Sharif University of Technology, P.O. Box 3406, *Tehran, Iran*

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We shall consider the second-order scalar differential equation of the form

$$x'' + f(x, x') = 0, \quad ' \equiv \frac{d}{dt}, \quad '' = \frac{d^2}{dt^2}$$

where f is continuous and smooth enough to ensure the existence and uniqueness of solution with any set of initial data. It will be shown if some conditions are satisfied, then there exists at least one nontrivial periodic solution. We also give some applications, for example the well known Levision-Smith equation.

Let us consider the following second-order differential equation

$$x'' + f(x, x') = 0, \quad ' \equiv \frac{d}{dt}, \quad '' \equiv \frac{d^2}{dt^2} \qquad \ldots(1)$$

where f is a continuous real-valued function with domain $\mathbf{R}^2$ and is smooth enough to ensure the existence and uniqueness of solution with any set of initial data.

Under the above assumptions, we shall establish the following theorem.

Theorem 1 — Let us assume that there exist constants α, β, $a > 0$ and $c_2 > 0$ such that

(i) $\alpha \leqq \beta$,

(ii) $f(\alpha, 0) \leqq 0 \leqq f(\beta, 0)$,

(iii) $c_2 > d = \text{Max}\{|\alpha|, |\beta|\}$,

(iv) $3M < a(c_2 - d)$

where $\quad M = \text{Max}\{|ax - f(x, x')| : (x, x') \in A\}$

and $\quad A = \{(x, x') \in \mathbf{R}^2 : |x| \leqq 2c_2, |x'| \leqq 2c_2\sqrt{a}\}$.

Then there exists at least an $\omega \in \left(\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}}\right)$ such that (1) has a nontrivial solution satisfying the following boundary conditions:

$$x(0) = x(\omega) \qquad \ldots(2)$$

$$x'(0) = x'(\omega). \qquad \ldots(2a)$$

PROOF: For each $\alpha \leqq c_1 \leqq \beta$, let us define the function $x(t) = x(t; c_1)$ as the solution of the following integral equation

$$x(t) = c_1 \cos a^{1/2} t + c_2' \sin a^{1/2} t + F(t, x(t), x'(t)) \qquad \ldots(3)$$

where $$F(t, x(t), x'(t)) = a^{-1/2} \int_0^t \sin a^{1/2}(t - s) \{ax(s) - f(x(s), x'(s))\}\, ds$$

and $$c_2' = c_2 \,.\, \text{sign}\, c_1.$$

One can easily verify that $x(t)$ satisfies eqn. (1), and furthermore $x(t)$ is not identically constant, since the velocity at the time $t = 0$ is equal to $c_2' \sqrt{a}$ which is not zero.

Now, let $B = \left\{x(t) \in C^1\left[0, \frac{3\pi}{2\sqrt{a}}\right] : |x(t)| \leqq 2c_2,\ |x'(t)| \leqq 2c_2\sqrt{a}\right\}$, and define the operator U on B by

$$Ux(t) = c_1 \cos a^{1/2} t + c_2' \sin a^{1/2} t + F(t, x(t), x'(t))$$

Then we have the following estimations

$$|Ux(t)| \leqq |c_1| + |c_2| + 3Ma^{-1},$$

and $$|(Ux)'(t)| \leqq a^{1/2}(|c_1| + |c_2| + 3Ma^{-1}).$$

Hence, by virtue of (iv), U maps B continuously into itself and it follows from Schauder's fixed point theorem that (2) has a solution in B. Now, since for any $x \in B$, $|F(t, x, x')| \leqq 3M/a$, one obtains

$$[c_1 - c_2' - F(\pi/2\sqrt{a}, x(\pi/2\sqrt{a}), x'(\pi/2\sqrt{a}))]$$

$$\times [c_1 + c_2' - F(3\pi/2\sqrt{a}, x(3\pi/2\sqrt{a}), x'(3\pi/2\sqrt{a}))] < 0$$

which leads to the existence of an $\omega \in (\pi/2\sqrt{a}, 3\pi/2\sqrt{a})$ such that

$$c_1(1 - \cos a^{1/2}\omega) - c_2 \sin a^{1/2}\omega = F(\omega, x(\omega), x'(\omega))$$

or (2).

Now, for each c_1, $\alpha \leqq c_1 \leqq \beta$, using Arzela-Ascoli theorem, one can easily verify that this solution is a C^1-function on $[0, \omega] \times [\alpha, \beta]$. Furthermore we claim that

$$x(t; \beta) \geqq \beta \quad \text{and} \quad x(t; \alpha) \leqq \alpha \quad \text{for} \quad 0 \leqq t \leqq \omega. \qquad \ldots(4)$$

In fact, let $\epsilon > 0$ be given and define

$$F_\epsilon(x, x') = \begin{cases} f(x, x') & \text{if } x \geqq \beta \\ f(\beta, x') - \epsilon(x - \beta) & \text{if } x < \beta. \end{cases}$$

Taking M, c_1, c_2 and $a > 0$ as above, we repeat the first part of the proof for the modified equation

$$x'' + F_\epsilon(x, x') = 0 \qquad \ldots(5)$$

it follows that there exists $\omega_\epsilon \in (\pi/2\sqrt{a}, 3\pi/2\sqrt{a})$ such that (5) possesses a non-trivial solution, say $x_\epsilon(t; c_1)$, such that

$$x_\epsilon(0; c_1) = x_\epsilon(\omega_\epsilon; c_1) = c_1.$$

Now, we would like to show that

$$x_\epsilon(t; \beta) \geqq \beta \quad \text{for} \quad 0 \leqq t \leqq \omega_\epsilon \qquad \ldots(6)$$

Suppose (6) is not true; i.e. there exists $t_1 \in [0, \omega_\epsilon]$ such that

$$x_\epsilon(t_1; \beta) - \beta < 0.$$

Since $x_\epsilon(0; \beta) = x_\epsilon(\omega_\epsilon; \beta) = \beta$, it follows that $x_\epsilon(t; \beta) - \beta$ must have a negative minimum at $t_0 \in [0, \omega_\epsilon]$, and hence $x'_\epsilon(t_0; \beta) = 0$. But

$$x''_\epsilon(t_0; \beta) = -f(\beta, 0) + \epsilon(x_\epsilon(t_0; \beta) - \beta) < 0$$

which contradicts the fact that the solution have a minimum at t_0. Hence (6) is true.

With a similar argument we can prove that there exists $x_\epsilon(t; \alpha)$ solution of the other modified equation $x'' + G_\epsilon(x, x') = 0$, such that $x_\epsilon(t; \alpha) \leqq \alpha$.

Now, from the definition of $F_\epsilon(x, x')$, we observe that ϵ may be chosen such that $\omega_\epsilon \to \omega$ as $\epsilon \to 0$, and that $x(t; \beta)$ and $x(t; \alpha)$ may be obtained as the limits of these solutions satisfying (4).

Finally, define the continuous function g on $[\alpha, \beta]$ by $g(c) = x'(0; c) - x'(\omega; c)$ we conclude by (4) that $g(\alpha) \leqq 0 \leqq g(\beta)$. Hence there must be a constant $\bar{c}$, $\bar{c} \in [\alpha, \beta]$ such that $g(\bar{c}) = 0$. This complets the proof of the theorem, by remarking that the solution $x(t; \bar{c})$ satisfies the periodic boundary conditions (2) and (2*a*).

Applications

(1) As a first application of the above theorem, consider the Levinson-Smith (1949) equation

$$x'' + f(x, x')\, x' + g(x) = 0. \qquad \ldots(7)$$

Theorem 2 — Let f and g be two functions of class C^2 in a neighbourhood of the origins of $\mathbf{R}^2$ and $\mathbf{R}$, satisfying the following conditions:

(i) $f(0, x') = 0$,

(ii) $g(x)$ is an odd function of x,

(iii) There exists a positive constant a such that $g'(0) = a^2$.

Then there exists at least one ω-periodic solution for (7), for some

$\omega \in (\pi/2a, 3\pi/2a)$.

PROOF: By the mean value theorem, there are constants c_3 and K_1 such that

$$| a^2x - g(x) | \leqq K_1 | x |^2 \quad \text{for} \quad | x | \leqq 2c_3.$$

and furthermore, since $g(x)$ is an odd function of x, there exists a positive constant ϵ such that $g(-\epsilon) \leqq 0 \leqq g(\epsilon)$.

Now, since $f(0, x') = 0$, applying again the mean value theorem we can prove the existence of the constants c_4 and K_2 such that

$$| f(x, x') | \leqq K_2 | x | \quad \text{for} \quad | x | \leqq 2c_4 \quad \text{and} \quad | x' | \leqq 2c_4 a.$$

Hence if $c_2 = \text{Min}\,\{c_3, c_4\}$, then we have for $| x | \leqq 2c_2$ and $| x' | \leqq 2c_2 a$

$$M = | a^2x - f(x, x')\, x' - g(x) | \leqq 4K_1c_2^2 + 4K_2ac_2^2 \leqq a^2(c_2 - \epsilon)/3$$

and Theorem 1 can be applied provided c_2 and ϵ are sufficiently small.

(2) Next let us consider the following equation

$$x'' + f(x)\, x'^n + g(x) = 0, \qquad n > 2. \qquad \ldots(8)$$

Theorem 3 — Let f and g by two functions of class C^2 in a neighbourhood of the origin in $\mathbf{R}$, and satisfying the following conditions

(*i*) $g(x)$ is an odd function of

(*ii*) For some $a > 0$, $g'(0) = a^2$.

Then there exists at least one ω-periodic solution for (8), for some

$\omega \in (\pi/2a, 3\pi/2a)$.

PROOF: A similar computation as in Theorem 2 yields

$$M = | ax - f(x)\, x'^n - g(x) | \leqq 4 \,|\, K_1 + M_1(2a)^n \,.\, c_2^{n-2} \,|\, c_2^2$$

for $| x | \leqq 2c_2$, $| x' | \leqq 2c_2 a$ and $M_1 = \text{Max}\, | f(x) |$, for $| x | \leqq 2c_2$. Then obviously we have

$$12 \,|\, K_1 + M_1(2a)^n \,.\, c_2^{n-2} \,|\, c_2^2 \leqq a^2(c_2 - \epsilon)$$

provided c_2 and ϵ are sufficiently small.

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