Indian J. pure appl. Math., 12(6) : 685-690, June 1981

REMARKS ON NEARLY COMPACT SPACES

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(*Received* 1 *May* 1980)

By a careful study of near compactness, the concept "(X, τ) is a nearly compact space" should never have been defined because this is just another way to say that (X, τ^*) is compact, where τ^* is the semi-regular topology associated with τ. Thus, in this paper, we show that theorems about nearly compact (resp. nearly paracompact, or nearly strongly paracompact) follow easily from known results about compact (resp. paracompact, or strongly paracompact) spaces. It is proved also that a paracompact subset of a Hausdorff space is actually weakly closed.

Singhal and Mathur (1969) have introduced the notion of nearly compact spaces. These are characterized by the property that "every open cover of the space has a finite subcollection, the interiors of the closures of which cover the space. The notion of nearly paracompact spaces was introduced by Singhal and Arya (1969a) as follows: a space is said to be nearly paracompact if every open cover of the space admits a locally-finite family of open subsets which refines it and the family of the interiors of the closures of whose members covers the space. In another paper Smirnov (1956) has introduced a class of spaces called strongly paracompact spaces. A spaces is said to be strongly paracompact if every open cover of the space has a star finite open refinement. Spaces with the property that for every open cover of the space there exists a star finite family of open sets which refines it and the interiors of the closurs of whose members cover the space are called nearly strongly paracompact spaces due to Kovačević (1978b).

By a careful study of nearly compactness, we think that the concept "(X, τ) is nearly compact (nearly paracompact, or nearly strongly paracompact)" should never have been defined because this is just another way to say that" (X, τ^*) is compact (paracompact, or strongly paracompact)" where τ^* is the semi-regular topology (Singhal and Mathur 1974) associated with τ. Thus, theorems about nearly compact (nearly paracompact, or nearly strongly paracompact) spaces follow easily from known results about compact (paracompact, or strongly paracompact) spaces.

In the present paper, we try to prove and explain this point of view, by giving very simple proofs of some known results. Also, by the idea of semi-regular topology, we give new properties of some existing topological notions. It is known that in a Hausdorff space every paracompact subset is closed, we prove in Theorem 3.4 that a paracompact subset of a Hausdorff space is actually weakly closed.

Note : Most of the notations in this work are standard.

1. Definitions and Preliminaries

In a space X, a point $p \in X$ is called a weak limit point of A (Al-Taha and Naoum 1974), if the closure of any open neighbourhood of $p \in X$ meets A in a point other than p. The set A'' of all weak limit points of A is called the weak derived set of A (Al-Taha and Naoum 1974). If $A'' \subset A$, then A is called weakly closed. Also, A is called strongly open if for every point x of A, there exists an open neighbourhood N_x of x such that its closure is contained in A. If $A = A^{0-}$, then A is called regularly closed. It is clear that, weak closedness implies closedness and strongly openness implies openness. In general, weak closedness and regularly closedness are independent properties. A subset $A \subset X$ is called nearly compact if for every cover of A by regularly open sets of X, there exists a finite subcover of A.

A subset $A \subset X$ is called a-nearly paracompact if every cover of A by open subsets of X has a locally finite family which refines it and the interiors of the closures of whose members cover A. A is nearly paracompact if it is nearly paracompact as a subspace.

A space X is called semi-regular (Singhal and Mathur 1974) if the family of regularly open subsets of X forms a base for the topology on X. X is said to be a T_2'-space (Mashhour 1970) (Urysohn space) if for any two distinct points of X, there exist two disjoint closed neighbourhoods of them. It is called almost regular (Singhal and Arya 1978b) if any regularly closed set $F \subset X$ and any singleton $\{x\}$ disjoint from F can be strongly separated. If any two disjoint weakly closed subsets of X are strongly separated, then X is called weakly normal (Mashhour *et al.* 1980).

A mapping $f: X \to Y$, where X and Y are topological spaces, is called almost continuous in the sense of Singhal and Singhal (1968) (briefly a.c.S.) if the inverse image of every regularly open subset of Y is open in X. It is called almost open in the sense of Singhal and Singhal (briefly a.c.S.) if the image of each regularly open subset of X is open in Y.

2. On Nearly Compactness

It is clear that, a topological space (X, τ) is nearly compact iff (X, τ^*) is compact.

Lemma 2.1 — A weakly closed (a regularly closed) subset of (X, τ) is closed in (X, τ^*).

PROOF : Let A be a weakly closed subset of (X, τ), then $X - A$ is τ-strongly open, i.e., for each $x \in (X - A)$, there is an open neighbourhood N_x of x such that $x \in N_x \subset N_x^- \subset X - A$. Thus, $X - A \supset N_x^{-0}$ and so,

$$X - A = \cup \{N_x^{-0} : x \in X - A\}$$

which is τ^*-open. Hence A is τ^*-closed. Also, if A is τ-regularly closed in X, then A is τ^*-closed.

Lemma 2.2 — A topological space (X, τ) is Hausdorff iff (X, τ^*) is Hausdorff.

PROOF : Let $x, y \in X$ be two distinct points, then there exist two disjoint τ-open neighbourhoods N_x, N_y of x, y, respectively, and so there exist two disjoint regularly open neighbourhoods N_x^{-0}, N_y^{-0} of x and y, respectively. Hence (X, τ^*) is Hausdorff. Since $\tau^* \subset \tau$, the converse is obvious.

Theorem 2.1 (Singhal and Mathur 1969) — Every nearly-compact Hausdorff space (X, τ) is almost regular.

PROOF : It is obvious by Lemma 2.1 and Lemma 2.2.

Theorem 2.2 (Noiri 1974) — Every pair of disjoint nearly compact subsets of a Hausdorff space (X, τ) have disjoint regularly open neighbourhoods.

PROOF : Let A and B be nearly compact subsets of (X, τ), then A and B are τ^*-closed. Thus, there exist two disjoint τ^*-open neighbourhoods U and V of A and B with $A \subset U \subset U^{-0}$ and $B \subset V \subset V^{-0}$, $U^{-0} \cap V^{-0} = \phi$, i.e., U^{-0} and V^{-0} are disjoint regularly open sets containing A and B.

Theorem 2.3 (Noiri 1974) — Each pair of disjoint nearly compact subsets of a T_2'-space have disjoint closed neighbourhoods.

PROOF : It is similar to that of the above theorem.

Theorem 2.4 (Mashhour *et al.* 1980) — Every weakly closed (regularly closed) subset A of a nearly compact space (X, τ) is nearly compact.

PROOF : Since A is weakly-closed in (X, τ), then by Lemma 2.1 A is τ^*-closed, and so, it is τ^*-compact. Thus, A is τ-nearly compact.

Theorem 2.5 (Mashhour *et al.* 1980) — Every nearly compact subset of a Hausdorff space (X, τ) is weakly closed.

PROOF : Let $A \subset X$ be a τ^*-compact. So, A and $X - A$ are τ^*-weakly closed and strongly open, respectively. This implies that $X - A$ is τ-strongly open. Hence, A is τ-weakly closed.

The following theorem gives an important property for almost continuous mappings in the sense of Singhal and Singhal (1968).

Theorem 2.6 — The following statements are equivalent for a mapping

$$f : (X, \tau_1) \to (Y, \tau_2) :$$

(i) f is almost continuous in the sense of Singhal and Singhal.

(ii) $f : (X, \tau_1) \to (Y, \tau_2^*)$ is continuous.

PROOF : Let $V \in \tau_2^*$, then for every $x \in V$, there exists a basic open τ_2^*-open neighbourhood N_x of x such that $N_x \subset V$. So, $V = \cup N_x$, then

$$f^{-1}(V) = f^{-1}(\cup N_x) = \cup f^{-1}(N_x)$$

and since $f^{-1}(N_x)$ is open in (X, τ_1), hence $f^{-1}(V)$ is open in (X, τ_1). Conversely, let V be a τ_2-regularly open, then V is τ_2^*-open, so $f^{-1}(V)$ is τ_1-open. Hence, f is almost continuous in the sense of Singhal and Singhal.

By using this property one can establish simple proofs for some theorems (cf. the proof of Theorem 2.7) on this type of continuity.

Theorem 2.7 (Singhal and Mathur 1969) — The image of compact space under almost continuous mapping in the sense of Singhal and Singhal is nearly compact.

PROOF : Let $f : (X, \tau_1) \to (Y, \tau_2)$ be almost continuous in the sense of Singhal and Singhal and surjection and let (X, τ_1) be compact. Then, $f : (X, \tau_1) \to (Y, \tau_2^*)$ is continuous and so, (Y, τ_2^*) is compact. Therefore, (Y, τ_2) is nearly compact.

3. On Nearly Paracompact and Nearly Strongly Paracompact Spaces

A topological space (X, τ) is:

(i) nearly paracompact iff (X, τ^*) is paracompact (Noiri 1977)

(ii) nearly strongly paracompact iff (X, τ^*) is strongly paracompact (Kovačević 1978a).

Theorem 3.1 (Kovačević 1978b) — Every nearly strongly paracompact Hausdorff space is almost regular.

PROOF : It is similar to that of Theorem 2.1.

Remark 3.1 : One can show that Theorems 2.1 and 3.1 are special cases of the following.

Theorem 3.2 — Let (X, τ) be a topological space such that (X, τ^*) is regular, then (X, τ) is almost regular.

PROOF : It is obvious.

Theorem 3.3 (Mashhour *et al.* 1980) — Let (X, τ) be a nearly paracompact and $A \subset X$ be a τ-weakly closed (regularly closed) subset, then A is τ-α-nearly paracompact.

PROOF : Let A be a τ^*-paracompact subset† and U be a τ-regularly open cover of A, then U is a basic τ^*-open cover of A, so there exists a locally finite τ^*-open refinement $V = \{V_j : j \in J\}$ of U. Then, V is a locally finite τ-open refinement of U. Hence, A is τ-α-nearly paracompact.

The following theorem gives a new property of paracompactness.

Theorem 3.4 — Every paracompact subset of a Hausdorff space is weakly closed.

PROOF : Let $A = \{y_i : i \in I\}$ be a paracompact subset of a Hausdorff space X. Without loss of generality, we may consider A as a proper subset of X. Take any point $x \in X - A$, then $x \neq y_i$, for each $i \in I$. So, there exist two families of open neighbourhoods : $U = \{U_i : x \in U_i, i \in I\}$ and $V = \{V_i : y_i \in V_i, i \in I\}$ of x and y_i, respectively, such that $U_i \cap V_i = \phi$, for each $i \in I$. One may consider V as an open cover of A, and since A is paracompact, then there exists a locally finite family $V^* = \{V_j : j \in J\}$ of open sets refining V and $A \subset \bigcup_{j \in J} V_j$. Therefore, there exists an open neighbourhood N_x of x intersects with a finite members $V_{j1}, V_{j2}, \ldots, V_{jn}$ of V^*. So, there exist the corresponding open neighbourhoods $U_1, U_2, \ldots, U_n$ of x such that $U_i \cap V_{ji} = \phi$, for all $i = 1, 2, \ldots, n$. Let $N_x \cap (\bigcap_{j=1}^{n} U_i) = N_x^*$, so $N_x^* \cap V_j = \phi$, for all $V_j \in V^*$, then $N_x^* \cap \{\bigcup V_j\} = \phi$, so $N_x^{*-} \cap (\bigcup V_j) = \phi$. Therefore, $N_x^{*-} \cap A = \phi$ and hence $X - A$ is strongly open, i.e., A is weakly closed.

Theorem 3.5 (Mashhour *et al.* 1980) — Let (X, τ) be a Hausdorff space and $A \subset X$ be a τ-α-nearly paracompact, then A is τ-weakly closed.

PROOF : Let $A \subset X$ be a τ^*-paracompact subset, so A is τ^*-weakly closed and this implies that $X - A$ is τ^*-strongly open. Hence, A is τ-weakly closed.

Theorem 3.6 (Mashhour *et al.* 1980) — Every nearly paracompact Hausdorff space (X, τ) is weakly normal.

PROOF : Let A, B be two disjoint τ-weakly closed subsets of X then A, B are τ^*-closed. Since (X, τ^*) is normal, (X, τ) is weakly normal.

Remark 3.2 : Indeed, the above theorem is a special case of the following.

Theorem 3.7 — Let (X, τ) be a topological space such that (X, τ^*) is normal, then (X, τ) is weakly normal.

†By a paracompact subset of X, we mean paracompact relative to X.

PROOF : It is obvious.

Theorem 3.8 (Kovačević 1978b) — If f is almost-continuous and open injection of a nearly strongly paracompact space (X, τ_1) onto a space (Y, τ_2) then (Y, τ_2) is nearly strongly paracompact.

PROOF : Let $\{U_i : i \in I\}$ be any basic τ_2^*-open cover of Y. Then $f^{-1}(U_i)$ is basic τ_1^*-open for each $i \in I$, since f is almost-continuous and open. Consider now, the basic τ_1^*-open cover $\{f^{-1}(U_i) : i \in I\}$ of X. Since, (X, τ_1^*) is strongly paracompact, there exists a star finite basic τ_1^*-open refinement $\{V_j : j \in J\}$ of $\{f^{-1}(U_i) : i \in I\}$. Thus $\{f(V_j) : j \in J\}$ is a star finite basic τ_2^*-open refinement of $\{U_i : i \in I\}$, and therefore, (Y, τ_2^*) is strongly paracompact. Hence, (Y, τ_2) is nearly strongly paracompact

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