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THE GENERATION THEOREM FOR THE MODIFIED SINE OPERATOR

RAMESH CHANDER AND A. B. BUCHE*

Department of Statistics, Panjab University, Chandigarh 160014

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In this paper we have defined the resolvent family associated with the modified sine operator on a Banach space and obtained the generation theorem for the modified sine operator.

1. INTRODUCTION

Let X be a Banach space, and let $B(X)$ be the space of bounded linear operators on X. Let $R^+ = [0, \infty)$. The family $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, is called a modified sine operator, if it satisfies the equation

$$U(t+s) + U(t-s) + 2U(t)\,U(s) = 2U(t)\,T(s) \qquad \ldots(1)$$

$s, t \in R^+$, $s \leqslant t$, $U(0) = 0$, where $\{T(t);\ t \in R^+\}$, $T : R^+ \to B(X)$, is a known (C_0)-semigroup of operators. We have proved (Ramesh Chander and Buche 1981b) that the continuity at the origin of $\{U(t)\}$ implies its continuity everywhere, and if $\{U(t)\}$ is continuous, then there exist two nonnegative constants M and ω such that

$$\| U(t) \| \leqslant M \exp(\omega t). \qquad \ldots(2)$$

Two infinitesimal generators of $\{U(t)\}$ have been introduced in another paper (see Ramesh Chander and Buche 1981a). The first infinitesimal generator of $\{U(t)\}$ is defined as

$$Ef = \lim_{h \to 0} (U(h)/h)\, f, \quad f \in D(E) \qquad \ldots(3)$$

where $D(E) \subset X$, and $D(E)$ is the set of elements $f \in X$ for which the above limit exists. The second infinitesimal generator F of $\{U(t)\}$ is defined by

$$Ff = \lim_{h \to 0} (2(T(h) - U(h) - I)/h^2)\, f \qquad \ldots(4)$$

$f \in D(F)$, where $D(F) \subset X$, and $D(F)$ is the set of elements $f \in X$ for which the above limit exists. It has been proved that $Ff = A^2 f = E^2 f$, $Af = Ef$, for $f \in D(F)$, where A is the infinitesimal generator of the associated semigroup $\{T(t)\}$ of operators. It was also seen that F is closable. Further, for $f \in D(F)$, $U(t)\,f$ is twice differentiable and

$$(d^2 U(t)/dt^2)\, f = U(t)\, f - tEf \qquad \ldots(5)$$

holds in the strong operator topology.

*Present address : Department of Statistics, Nagpur University, Nagpur.

In section 2 of the present paper, the generation theorem for the regular modified sine operator is obtained, which is similar to that for semigroups obtained by Hille, Yosida, Phillips, Miyadera and Feller (ref. Hille and Phillips 1957) and it is also similar to that for the regular cosine operator function obtained by Sova (1966). The appropriate resolvent turns out to be $A(\lambda^2 I - A^2)^{-1}$, where A is the first infinitesimal generator of the associated semigroup of (C_0)-type, under the assumption that $E = A$ and $F = A^2$. In the proofs, we use the properties of Laplace transforms such as uniqueness and the Post-Widder inversion formula (cf. Doetsch 1950, Widder 1966). Throughout we have assumed that $U(t)\,T(s) = T(s)\,U(t)$, and $U(t)\,U(s) = U(s)\,U(t)$, for all $t, s \in R^+$.

Section 3 contains some examples.

2. The Generation Theorem

Lemma 1 — Let $\{U(t);\ t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator. Then there exists a real number ω_0 such that for $\lambda > \omega_0$,

$$Z(\lambda)\, f = \int_0^\infty e^{-\lambda p} U(p)\, f\, dp$$

is a bounded linear operator.

Also if $f \in D(E)$, then $Z(\lambda)\, f \in D(F)$, and

$$(\lambda^2 I - F)\, Z(\lambda)\, f = Ef, \quad \lambda > \omega_0.$$

Further, if $f \in D(F)$, then $Z(\lambda)\, (\lambda^2 I - F)\, f = Ef$, for $\lambda > \omega_0$.

Proof : Since $U(t)$ is a regular modified sine operator, there exists, by (2), two nonnegative constants M and ω_0 such that $\| U(t) \| \leqslant M \exp(\omega_0 t)$, for every $t \in R^+$. So, for $\lambda > \omega_0$ and $t_0 > 0$,

$$\left\| \int_0^{t_0} e^{-\lambda t} U(t)\, f\, dt \right\| \leqslant \int_0^{t_0} e^{-\lambda t} M e^{\omega_0 t} \| f \|\, dt$$

$$\leqslant M \int_0^\infty e^{-(\lambda - \omega_0)t} \| f \|\, dt$$

$$= M \| f \| / (\lambda - \omega_0).$$

Clearly $Z(\lambda)$ is a linear operator and it has been proved that $Z(\lambda)$ is bounded for $\lambda > \omega_0$.

Consider

$$F_h \int_0^\infty e^{-\lambda p} U(p)\, f\, dp$$

$$= \left(\frac{1}{h^2}\right) \int_0^\infty e^{-\lambda p}\, (2T(h)\, U(p)\, f - 2U(h)\, U(p)\, f - 2U(p)\, f)\, dp$$

(*equation continued on p.* 693)

$$= \left(\frac{1}{h^2}\right) \int_0^h e^{-\lambda p}(2T(h)\,U(p)\,f - 2U(h)\,U(p)\,f - 2U(p)\,f)\,dp$$

$$+ \left(\frac{1}{h^2}\right) \int_h^\infty e^{-\lambda p}(U(p+h)\,f + U(p-h)\,f - 2U(p)\,f)\,dp$$

$$= \left(\frac{1}{h^2}\right)\left(2\int_0^h e^{-\lambda p}T(h)\,U(p)\,f\,dp - \int_0^h e^{-\lambda p}(2U(h)\,U(p)\,f)\,dp\right)$$

$$+ \left(\frac{1}{h^2}\right)\left(\int_0^\infty e^{-\lambda(p-h)}U(p)\,f\,dp - \int_0^{2h} e^{-\lambda(p-h)}U(p)\,f\,dp\right.$$

$$\left. + \int_0^\infty e^{-\lambda(p+h)}U(p)\,f\,dp - 2\int_0^\infty e^{-\lambda p}U(p)\,f\,dp\right)$$

$$= ((e^{\lambda h} + e^{-\lambda h} - 2)/h^2)\,Z(\lambda)\,f - \left(\frac{2}{h^2}\right) U(h) \int_0^h e^{-\lambda p}T(p)\,f\,dp$$

$$+ \left(\frac{2}{h^2}\right) T(h) \int_0^h e^{-\lambda p}U(p)\,f\,dp + \left(\frac{1}{h^2}\right)\left(e^{-\lambda h}\int_0^h e^{-\lambda p}U(p)\,f\,dp\right.$$

$$\left. - e^{\lambda h}\int_0^h e^{-\lambda p}U(p)\,f\,dp\right)$$

$$= J_1 - J_2 + J_3 - J_4, \text{ say.}$$

Clearly

$$J_1 \to \lambda^2 Z(\lambda)\,f,\ f \in X,\ h \to 0,\ h > 0.$$

Now

$$J_4 = \left(\frac{1}{h^2}\right)(e^{-\lambda h} - e^{\lambda h}) \int_0^h e^{-\lambda p}U(p)\,f\,dp$$

$$+ \left(\frac{1}{h^2}\right) e^{-\lambda h} \int_0^h (e^{\lambda p} - e^{-\lambda p})\,U(p)\,f\,dp$$

(*equation continued on p. 694*)

$$= ((e^{-\lambda h} - e^{\lambda h})/h)\left(\frac{1}{h}\right)\int_0^h e^{-\lambda p}U(p)f\,dp$$

$$+ (e^{-\lambda h}/h^2)\int_0^h (e^{\lambda p} - e^{-\lambda p})\, U(p)f\,dp.$$

Since

(i) $\lim_{\substack{h\to 0\\ h>0}} (e^{-\lambda h} - e^{\lambda h})/h = -2,$

(ii) $\left\|\left(\frac{1}{h}\right)\int_0^h e^{-\lambda p}U(p)f\,dp\right\| \leqslant \sup_{0\leqslant p\leqslant h} \| U(p) \|,$

(iii) $$\| (e^{-\lambda h}/h^2)\int_0^h (e^{\lambda p} - e^{-\lambda p})\, U(p)f\,dp \|$$

$$\leqslant \sup_{0\leqslant p\leqslant h} \| U(p)f \| \,(e^{-\lambda h}\int_0^h (e^{\lambda p} - e^{-\lambda p})\,dp/h^2)$$
$$= \sup_{0\leqslant p\leqslant h} \| U(p)f \| \,(e^{-\lambda h}(e^{\lambda h} + e^{-\lambda h} - 2)/\lambda h^2),$$

(iv) $\sup_{0\leqslant p\leqslant h} \| U(p)f\| \to 0,$ as $h \to 0,\ h > 0,$

it follows that

$$J_4 \to 0, \quad \text{as} \quad h \to 0,\ h > 0, \quad \text{for} \quad f \in X.$$

Now

$$J_2 = (2U(h)/h)\,((\int_0^h e^{-\lambda p}T(p)f\,dp)/h).$$

Since $\{T(t)\}$ is a semigroup of class-(C_0), it follows that

$$J_2 \to 2Ef, \quad \text{as} \quad h \to 0,\ h > 0, \quad \text{for} \quad f \in D(E).$$

Lastly,

$$J_3 - Ef = ((2T(h)\int_0^h e^{-\lambda p}U(p)f\,dp)/h^2) - Ef$$

$$= ((2T(h)\int_0^h e^{-\lambda p}(U(p)f - pEf)\,dp)/h^2$$

$$+ ((2T(h)\int_0^h e^{-\lambda p}pEf\,dp)/h^2) - Ef$$

(*equation continued on p.* 695)

$$= ((2T(h) \int_0^h e^{-\lambda p}(U(p)f - pEf)\, dp)/h^2)$$

$$+ (((2 \int_0^h pe^{-\lambda p}dp)/h^2) - 1)\, T(h)\, Ef + (T(h)\, Ef - Ef).$$

Since (i) $\{T(t)\}$ is of class-(C_0), (ii) $((2 \int_0^h pe^{-\lambda p}dp)/h^2) \to 1$, as $h \to 0$, $h > 0$, (iii) for $f \in D(E)$, one can find for $\epsilon > 0$, an $h > 0$ such that $\| (U(p)/p) f - Ef \| \leqslant \epsilon$, for $0 < p \leqslant h$, and (ii) holds, it follows that

$$J_3 - Ef \to 0, \quad \text{as} \quad h \to 0, \; h > 0 \quad \text{for} \quad f \in D(E).$$

Hence

$$\lim_{\substack{h \to 0 \\ h > 0}} F_h \int_0^\infty e^{-\lambda p} U(p) f\, dp$$

$$= \lambda^2 \int_0^\infty e^{-\lambda p} U(p) f\, dp - 2Ef + Ef \quad \text{for} \quad f \in D(E).$$

or

$$\int_0^\infty e^{-\lambda p} U(p) f\, dp \in D(F) \quad \text{for} \quad f \in D(E),$$

and

$$F \int_0^\infty e^{-\lambda p} U(p) f\, dp = \lambda^2 \int_0^\infty e^{-\lambda p} U(p) f\, dp - Ef,$$

or

$$FZ(\lambda) f = \lambda^2 Z(\lambda) f - Ef \quad \text{for} \quad f \in D(E).$$

Hence we have proved that

$$(\lambda^2 I - F)\, Z(\lambda) f = Ef, \quad \text{for} \quad f \in D(E).$$

If $f \in D(F)$, then $FZ(\lambda) f = Z(\lambda)\, Ff$. Also $f \in D(F)$ implies that $f \in D(E)$ and hence $Z(\lambda)\, Ff = \lambda^2 Z(\lambda) f - Ef$, by the above result, that is,

$$Z(\lambda)\, (\lambda^2 I - F) f = Ef \quad \text{for} \quad f \in D(F).$$

Convention : In what follows we shall assume that $E = A$ and $F = A^2$.

Proposition 1 — Let $\{U(t);\, t \in R^+\}$, $U : R^+ \to B(X)$, be a regular modified sine operator with the infinitesimal generators $E = A$ and $F = A^2$, where A is the infinitesimal generator of the associated semigroup $\{T(t)\}$. Then there exists an $\omega_0 > 0$, such that for $\lambda > \omega_0$,

$$Z(\lambda) f = \int_0^\infty e^{-\lambda p} U(p) f \, dp = A(\lambda^2 I - A^2)^{-1} f,$$

$f \in X$.

PROOF: Since the closed linear operator A is the infinitesimal generator of the semigroup $\{T(t)\}$ of class-(C_0), it follows that there exists $\omega_0 > 0$, such that $(\lambda I - A)^{-1} \in B(X)$, for $\lambda > \omega_0$. Since A^2 is a polynomial in A, and A^2 is a closed linear operator, (cf. Hille and Phillips 1957), it follows by the spectral mapping theorem, (cf. Dunford and Schwartz 1958), that $(\lambda^2 I - A^2)^{-1} \in B(X)$ for $\lambda^2 > \omega_0^2$, that is, $\lambda > \omega_0$. Thus by the Lemma 1,

$$(\lambda^2 I - A^2) Z(\lambda) f = Af, \quad f \in D(A),$$

and

$$Z(\lambda) (\lambda^2 I - A^2) f = Af, \quad f \in D(A^2).$$

Hence $Z(\lambda) f = A(\lambda^2 I - A^2)^{-1} f = (\lambda^2 I - A^2)^{-1} Af$, on $D(A^2)$, the last two being equal because A and $(\lambda^2 I - A^2)^{-1}$ commute on $D(A^2)$. Now, for $f \in X$,

$$\begin{aligned} R(\lambda, A) f - \lambda(\lambda^2 I - A^2)^{-1} f &= (R(\lambda, A)(\lambda^2 I - A^2) - \lambda I)(\lambda^2 I - A^2)^{-1} f \\ &= (R(\lambda, A)(\lambda I - A)(\lambda I + A) - \lambda I)(\lambda^2 I - A^2)^{-1} f \\ &= (\lambda I + A - \lambda I)(\lambda^2 I - A^2)^{-1} f \\ &= A(\lambda^2 I - A^2)^{-1} f. \end{aligned}$$

Therefore, we have seen that $A(\lambda^2 I - A^2)^{-1}$ can be expressed as the difference of two linear bounded operators, and hence $A(\lambda^2 I - A^2)^{-1}$ is a linear bounded operator. We also know that if two linear bounded operators agree on a dense set, then the two operators are equal. Thus

$$Z(\lambda) f = A(\lambda^2 I - A^2)^{-1} f, \quad \text{for} \quad f \in X.$$

Proposition 2 — Under the hypothesis of the Proposition 1, there exists an $\omega > 0$ such that for $\lambda > \omega$ and $n = 0, 1, 2, \ldots, f \in X$,

$$\| (d^n A(\lambda^2 I - A^2)^{-1} / d\lambda^n) f \| \leqslant M(n!/2) ((1/(\lambda + \omega)^{n+1}) + (1/(\lambda - \omega)^{n+1})) \| f \|.$$

PROOF: In the notations of Corollary 1 of Ramesh Chander and Buche (1981b) and by the properties of Laplace transforms (Widder 1966), and usingl inear functional, we have

$$\| (d^n Z(\lambda)/d\lambda^n) f \|$$
$$= \| (d^n(\int_0^\infty e^{-\lambda p} U(p) f \, dp)/d\lambda^n) \|$$
$$= \| \int_0^\infty e^{-\lambda p} p^n U(p) f \, dp \|$$
$$\leqslant M \| f \| \int_0^\infty e^{-\lambda p} p^n \cosh (\omega p) \, dp$$
$$= M(n!/2) ((1/(\lambda + \omega)^{n+1} + (1/(\lambda - \omega)^{n+1})) \| f \|.$$

Theorem — Let $\{T(t); t \in R^+\}$, $T : R^+ \to B(X)$, be a semigroup of class-(C_0). Let A be the infinitesimal generator of $\{T(t)\}$. If there exist two nonnegative constants M and ω such that

(i) $\| T(t) \| \leqslant M \exp (\omega t)$, $t \in R^+$,

(ii) for each $\lambda > \omega$, $A(\lambda^2 I - A^2)^{-1} \in B(X)$,

(iii) for every $\lambda > \omega$ and every $n = 0, 1, 2, \ldots$,

$$\| (d^n A(\lambda^2 I - A^2)^{-1}/d\lambda^n) f \| \leqslant M(n!/2) ((1/(\lambda + \omega)^{n+1}) + (1/(\lambda - \omega)^{n+1})) \| f \|,$$

then there exists a regular modified sine operator $\{U(t); t \in R^+\}$, $U : R^+ \to B(X)$, with the first and second infinitesimal generators E and F such that

(1) $\bar{E} = A$ and $\bar{F} = A^2$,

(2) for every $t \in R^+$, $\| U(t) \| \leqslant M \cosh (\omega t)$.

PROOF : The theorem will be proved in parts.

1° If $\lambda > \omega$, $f \in X$, $l \in X^*$, then we write $\pi(f, l)(\lambda) = l(A(\lambda^2 I - A^2)^{-1} f)$. Further we write

$$\pi^{(n)}(f, l)(\lambda) = (d^n \pi(f, l)(\lambda)/d\lambda^n),$$

for every $n = 0, 1, 2, \ldots$.

Then for $\lambda > \omega$, $f \in X$ and $l \in X^*$ and $n = 0, 1, 2, \ldots$, we have, in view of the condition (iii),

$$\begin{aligned} | \pi^{(n)}(f, l)(\lambda) | &= | d^n l(A(\lambda^2 I - A^2)^{-1} f/d\lambda^n | \\ &\leqslant \| l \| \, \| d^n A(\lambda^2 I - A^2)^{-1} f/d\lambda^n \| \\ &\leqslant M(n!/2) ((1/(\lambda + \omega)^{n+1}) + (1/(\lambda - \omega)^{n+1})) \| f \| \, \| l \| \\ &\leqslant M(n!/(\lambda - \omega)^{n+1}) \| f \| \, \| l \|. \end{aligned}$$

2^0 For each $f \in X$ and $l \in X^*$ there exists a real measurable function φ on R^+ such that

(1) for every $t \in R^+$, $|\varphi(t)| \leqslant M \exp(\omega t) \| f \| \| l \|$,

(2) for every $\lambda > \omega$, we have

$$\int_0^\infty e^{-\lambda p}\varphi(p)\, dp = \pi(f, l)(\lambda).$$

We denote this φ by $\Phi(f, l)$.

PROOF: Let $f \in X$, $l \in X^*$ be fixed. Define, for $\nu > 0$, the function

$$g(\nu) = \pi(f, l)(\nu + \omega).$$

Then evidently by 1^0

$$| d^n g(\nu)/d\nu^n | \leqslant M(n!/\nu^{n+1}) \| f \| \| l \|, \quad n = 0, 1, 2, \ldots .$$

Now we use Theorems 16(a) and 16(b) in Chapter VII of Widder (1966) and find that there exists a real measurable function φ_1 on R^+ such that

(i) $|\varphi_1(t)| \leqslant M \| f \| \| l \|$, $t \in R^+$, and

(ii) $g(\nu) = \int_0^\infty e^{-\nu p}\varphi_1(p)\, dp$, $\nu > 0$.

Now taking $\varphi(p) = \exp(\omega p)\, \varphi_1(p)$, we obtain the desired result.

3^0 If $f \in D(A^2)$ and $l \in X^*$, then we define a new function $\Psi(f, l)$: for $t \in R^+$,

$$\Psi(f, l)(t) = tl(Af) + \int_0^t \left(\int_0^s \Phi(A^2 f, l)(u)\, du\right) ds.$$

This definition is clearly meaningful by 2^0.

Assume, for convenience, that $t_2 > t_1$. Then for each $f \in D(A^2)$, $l \in X^*$, we have, by 2^0,

$$\begin{aligned}
&| \Psi(f, l)(t_2) - \Psi(f, l)(t_1) | \\
&\leqslant | t_2 - t_1 | \| l \| \| Af \| + \left| \int_{t_1}^{t_2} \left(\int_0^s \Phi(A^2 f, l)(u)\, du\right) ds \right| \\
&\leqslant | t_2 - t_1 | \| l \| \| Af \| + (t_2 - t_1) \sup_{t_1 \leqslant s \leqslant t_2} \left| \int_0^s \Phi(A^2 f, l)(u)\, du \right| \\
&\leqslant | t_2 - t_1 | \| l \| \| Af \| + | t_2 - t_1 | \sup_{t_1 \leqslant s \leqslant t_2} Mse^{\omega s} \| l \| \| A^2 f \| \\
&\leqslant | t_2 - t_1 | \| l \| \| Af \| + | t_2 - t_1 | M(t_1 + t_2) \exp(\omega(t_1 + t_2)) \| l \| \| A^2 f \|.
\end{aligned}$$

4^0 For each $f \in D(A^2)$, $l \in X^*$ and $\lambda > \omega$, we have

$$\int_0^\infty e^{-\lambda t}\Psi(f, l)(t)\, dt = \int_0^\infty e^{-\lambda t}\Phi(f, l)(t)\, dt.$$

PROOF : $\int_0^\infty e^{-\lambda t}\Psi(f, l)(t)\, dt$

$$= l(Af) \int_0^\infty e^{-\lambda t} t\, dt + \int_0^\infty e^{-\lambda t}\Big(\int_0^t \Big(\int_0^s \Phi(A^2 f, l)(u)\, du\Big)\, ds\Big)\, dt$$

$$= (l(Af)/\lambda^2) + \int_0^\infty \int_u^\infty \int_s^\infty e^{-\lambda t}\Phi(A^2 f, l)(u)\, dt\, ds\, du$$

$$= (l(Af)/\lambda^2) + (1/\lambda^2) \int_0^\infty e^{-\lambda u}\Phi(A^2 f, l)(u)\, du$$

$$= (l(Af)/\lambda^2) + (1/\lambda^2)\, \pi(A^2 f, l)(\lambda)$$

$$= (l(Af)/\lambda^2) + (l(A(\lambda^2 I - A^2)^{-1} A^2 f)/\lambda^2)$$

$$= l((1/\lambda^2)(Af + A((\lambda^2 I - A^2)^{-1} A^2 f)))$$

$$= l((1/\lambda^2)(Af + A(-(\lambda^2 I - A^2)^{-1}(\lambda^2 I - A^2) f + \lambda^2(\lambda^2 I - A^2)^{-1} f)))$$

$$= l(1/\lambda^2)(Af - Af + \lambda^2 A(\lambda^2 I - A^2)^{-1} f))$$

$$= l(A(\lambda^2 I - A^2)^{-1} f)$$

$$= \pi(f, l)(\lambda)$$

$$= \int_0^\infty e^{-\lambda t}\Phi(f, l)(t)\, dt.$$

5^0 For every $f \in D(A^2)$ and $l \in X^*$, there exists a null subset $N(f, l)$ of R^+, such that for all $t \in R^+ \setminus N(f, l)$, we have $\Psi(f, l)(t) = \Phi(f, l)(t)$.

PROOF : This is an immediate consequence of 4^0 and the properties of Laplace transforms (Widder 1966).

6^0 For every $f \in D(A^2)$, $l \in X^*$ and $t \in R^+$, we have

$$|\Psi(f, l)(t)| \leqslant M \exp(\omega t)\, \|f\|\, \|l\|.$$

PROOF : This is an easy consequence of 5^0, 2^0 and 3^0, since 3^0 evidently implies the continuity of $\Psi(f, l)(t)$ in $t \in R^+$.

7^0 For each $f \in D(A^2)$, $t > 0$, and $\epsilon > 0$, there exists a natural number $n_0(f, t, \epsilon)$ such that $n_0(f, t, \epsilon) > t\omega$, and for $n \geqslant n_0(f, t, \epsilon)$ and each $l \in X^*$,

$$|((-1)^n/n!)(n/t)^{n+1}\, \pi^{(n)}(f, l)(n/t) - \Psi(f, l)(t)| \leqslant \epsilon\, \|l\|.$$

PROOF : Let $S = \{l : l \in X^*, \| l \| \leqslant 1\}$ and $\Gamma(S)$ be the set of all bounded real functions on S. If $\varphi \in \Gamma(S)$, then we take $||| \varphi ||| = \sup_{l \in S} | \varphi(l) |$. Evidently $\Gamma(S)$ is a Banach space.

Now, let $f \in D(A^2)$ be fixed (all the objects considered in the following depend on f-this dependence is not marked explicitly). We define a function θ on R^+ with values in $\Gamma(S)$ as follows : if $l \in S$, then $\theta(t)(l) = \Psi(f, l)(t)$. For $\lambda > \omega$, we define a function $G(\lambda)(l) = \pi(f, l)(\lambda)$, $l \in S$.

Now by 3^0, θ is continuous on R^+ with respect to the topology of $\Gamma(S)$, and, further by 6^0, the following estimate is valid : if $t \in R^+$,

$$||| \theta(t) ||| \leqslant M \exp(\omega t) \| f \| \| l \|.$$

By the use of 2^0 and 4^0 we easily see that, for $\lambda > \omega$,

$$\int_0^\infty e^{-\lambda u}\theta(u)\, du = G(\lambda),$$

(where the integral is taken with respect to the topology of $\Gamma(S)$).

Now, by the Theorem 1, p. 130 of Doetsch (1950), we obtain (writing $d^n G(\lambda)/d\lambda^n = G^{(n)}(\lambda)$, $n = 0, 1, 2, \ldots,$)

$$((-1)^n/n!)\,(n/t)^{n+1}\, G^{(n)}(n/t) \to \theta(t),$$

with respect to the topology of $\Gamma(S)$, which is in fact the desired result.

8^0 For $\lambda > \omega$ and $n = 0, 1, 2, \ldots$, we define

$$V_n(\lambda) = ((-1)^n/n!)\,\lambda^{n+1}(d^n A(\lambda^2 I - A^2)^{-1}/d\lambda^n).$$

Then for each $f \in X$, $l \in X^*$ and $\lambda > \omega$, we have

$$((-1)^n/n!)\,\lambda^{n+1}\pi^{(n)}(f, l)(\lambda) = l(V_n(\lambda) f).$$

The proof is evident from 1^0.

9^0 For each $f \in D(A^2)$ and $t \in R^+$, the sequence $V_n(n/t) f$, $(n > t\omega)$, is a Cauchy sequence.

PROOF : Let $f \in D(A^2)$ and $t \in R^+$ be given, and let $\epsilon > 0$. Then by 8^0 and 7^0, there exists a natural number $n_0(t, f, \epsilon/2) > t\omega$ such that, for each $n > n_0(t, f, \epsilon/2)$ and $l \in X^*$,

$$| l(V_n(n/t) f) - \Psi(f, l)(t) | \leqslant (\epsilon/2) \| l \|.$$

Hence, for each $n_1, n_2 \geqslant n_0(t, f, \epsilon/2)$ and $l \in X^*$,

$$| l(V_{n_1}(n_1/t) f) - l(V_{n_2}(n_2/t) f) | \leqslant \epsilon \| l \|.$$

But this implies that

$$\| V_{n_1}(n_1/t) f - V_{n_2}(n_2/t) f \| \leqslant \epsilon.$$

10⁰ For each $t \in R^+$ we define an operator $U_0(t)$ on $D(A^2)$ with values in X as follows:

$$U_0(t) f = \lim_{\substack{n \to \infty \\ n > t\omega}} V_n(n/t) f,$$

for each $f \in D(A^2)$.

Then for each $f \in D(A^2)$, $l \in X^*$ and $t \in R^+$, we have $l(U(t) f) = \Psi(f, l)(t)$.

PROOF : Consequence of 8⁰ and 7⁰.

11⁰ For each $f \in D(A^2)$, the function $U_0(.) f$ is continuous on R^+, and $U_0(t) f \to 0$, as $t \to 0$, $t \in R^+$.

PROOF : By 10⁰ and 3⁰, we have for each $t_1, t_2 \in R^+$, $f \in D(A^2)$ and $l \in X^*$,

$$\begin{aligned} &| l(U_0(t_1) f - U_0(t_2) f) | \\ &\quad \leqslant | t_1 - t_2 | \, \| l \| \, \| Af \| \\ &\qquad + | t_1 - t_2 | \, M(t_1 + t_2) \exp(\omega(t_1 + t_2)) \, \| A^2 f \| \, \| l \|. \end{aligned}$$

Hence

$$\begin{aligned} \| U_0(t_1) f - U_0(t_2) f \| &\leqslant | t_1 - t_2 | \, \| Af \| \\ &\quad + | t_1 - t_2 | \, M(t_1 + t_2) \exp(\omega(t_1 + t_2)) \, \| A^2 f \|, \end{aligned}$$

which gives the continuity of $U_0(.) f$ on R^+, $f \in D(A^2)$.

By 10⁰ and 3⁰,

$$| l(U_0(t) f) | \leqslant t \, \| Af \| \, \| l \| + t^2 M \exp(\omega t) \, \| A^2 f \| \, \| l \|,$$

that is,

$$\| U_0(t) f \| \leqslant t \, \| Af \| + t^2 M \exp(\omega t) \, \| A^2 f \|,$$

and this implies that $U_0(t) f \to 0$, as $t \to 0$, $t \in R^+$, for $f \in D(A^2)$.

12⁰ If $f \in D(A^2)$ and $t \in R^+$, then

$$\| U_0(t) f \| \leqslant M \cosh(\omega t) \, \| f \|.$$

PROOF : By the use of assumption (iii) of Proposition 3, we obtain, for each $n > t\omega$,

$$\begin{aligned} \| V_n(n/t) f \| &\leqslant (M/2) \, ((n/t)^{n+1}/((n/t) + \omega)^{n+1}) \\ &\quad + ((n/t)^{n+1}/((n/t) - \omega)^{n+1})) \, \| f \|. \end{aligned}$$

It is easy to verify that the right hand side of this inequality tends to

$$M \cosh(\omega t) = (M/2)(\exp(\omega t) + \exp(-\omega t)),$$

as $n \to \infty$, and this implies the result.

13° If $f \in D(A^4)$, then

$$(d^2 U_0(t) f/dt^2) = U_0(t) A^2 f, \quad \text{a.e.}$$

PROOF: From 10°, we have for each $f \in D(A^2)$, $l \in X^*$ and $t \in R^+$,

$$\begin{aligned} l(U_0(t) f) &= \Psi(f, l)(t) \\ &= tl(Af) + \int_0^t \left(\int_0^s \Phi(A^2 f, l)(u)\, du\right) ds. \end{aligned}$$

From this we get

$$\begin{aligned} (d^2 l(U_0(t) f)/dt^2) &= l(d^2 U_0(t) f/dt^2) \\ &= \Phi(A^2 f, l)(t). \end{aligned}$$

If $f \in D(A^4)$, then $A^2 f \in D(A^2)$, and hence by 5°,

$$\Psi(A^2 f, l)(t) = \Phi(A^2 f, l)(t), \quad \text{a.e.}$$

Thus

$$\begin{aligned} l(d^2 U_0(t) f/dt^2) &= \Psi(A^2 f, l)(t) \\ &= l(U_0(t) A^2 f), \quad \text{a.e.}, \quad l \in X^*, \end{aligned}$$

that is,

$$(d^2 U_0(t) f/dt^2) = U_0(t) A^2 f, \quad \text{a.e.}$$

14° For every $f \in D(A^2)$ and $\lambda > \omega$, we have

$$\int_0^\infty e^{-\lambda p} U_0(p) f\, dp = A(\lambda^2 I - A^2)^{-1} f.$$

PROOF: The existence is an immediate consequence of 12° and 11°. Now it suffices to prove that

$$\int_0^\infty e^{-\lambda p} l(U_0(p) f)\, dp = l(A(\lambda^2 I - A^2)^{-1} f)$$

for every $l \in X^*$. But this follows from 10°, 4° and 1°.

15° $D(A^2)$ is dense in X.

PROOF: Since A is the infinitesimal generator of a regular semigroup, $D(A^2)$ is dense in X.

16^0 For each $t \in R^+$, there exists one and only one continuous extension of $U_0(t)$ to the whole space X. We denote this extension by $U(t)$. Also

$$\| U(t) \| \leqslant M \cosh (\omega t).$$

PROOF : A simple consequence of 11^0, 12^0 and 15^0.

17^0 For each $f \in X$, we have $U(t) f \to 0$, as $t \to 0$, $t \in R^+$.

PROOF : For $f \in D(A^2)$, we have 11^0, and then using 16^0 and the Banach-Steinhauss theorem, the result follows.

18^0 For every $f \in X$ and $\lambda > \omega$, we have

$$\int_0^\infty e^{-\lambda p} U(p) f \, dp = A(\lambda^2 I - A^2)^{-1} f.$$

PROOF : The proof follows from 14^0, 15^0 and 16^0 and the fact that $A(\lambda^2 I - A^2)^{-1}$ is a bounded linear operator.

19^0 Let $A_1(\lambda) = A(\lambda^2 I - A^2)^{-1} + A(\mu^2 I - A^2)^{-1}$, $A_2(\lambda) = A(\lambda^2 I - A^2)^{-1} - A(\mu^2 I - A^2)^{-1}$, for $\lambda > \omega$, $\mu > \omega$, $\lambda \neq \mu$, then since A, $(\lambda I - A)^{-1}$, $(\lambda^2 I - A^2)^{-1}$, $(\mu I - A)^{-1}$, $(\mu^2 I - A^2)^{-1}$ commute on $D(A^2)$, and since

$$\begin{aligned}(\lambda I + A)(\lambda^2 I - A^2)^{-1} &= (\lambda I - A)^{-1} (\lambda I - A)(\lambda I + A)(\lambda^2 I - A^2)^{-1} \\ &= (\lambda I - A)^{-1} (\lambda^2 I - A^2)(\lambda^2 I - A^2)^{-1} \\ &= (\lambda I - A)^{-1}\end{aligned}$$

and similarly $(\mu I + A)(\mu^2 I - A^2)^{-1} = (\mu I - A)^{-1}$, we have

$$\begin{aligned}&(1/(\lambda + \mu))(A(\lambda^2 I - A^2)^{-1} f + A(\mu^2 I - A^2)^{-1} f) - (1/(\lambda - \mu)) \\ &\quad \times (A(\lambda^2 I - A^2)^{-1} f - A(\mu^2 I - A^2)^{-1} f) - 2(\lambda I - A)^{-1} A(\mu^2 I - A^2)^{-1} f \\ &\quad - 2(\mu I - A)^{-1} A(\lambda^2 I - A^2)^{-1} f + (2/(\lambda + \mu)) A(\lambda I + \mu I - 2A)^{-1} \\ &\quad \times (\lambda I - A)^{-1} f + (2/(\lambda + \mu)) A(\lambda I + \mu I - 2A)^{-1} (\mu I - A)^{-1} f \\ &= (1/(\lambda + \mu)) A_1(\lambda) f - (1/(\lambda - \mu)) A_2(\lambda) f - 2(\lambda I - A)^{-1} A(\mu^2 I - A^2)^{-1} f \\ &\quad - 2(\mu I - A)^{-1} A(\lambda^2 I - A^2)^{-1} f + (2/(\lambda + \mu)) A(\lambda I + \mu I \\ &\quad - 2A)^{-1} ((\mu I - A) + (\lambda I - A))(\lambda I - A)^{-1} (\mu I - A)^{-1} f \\ &= (1/(\lambda + \mu)) A_1(\lambda) f - (1/(\lambda - \mu)) A_2(\lambda) f - 2(\lambda I - A)^{-1} A(\mu^2 I - A^2)^{-1} f \\ &\quad - 2(\mu I - A)^{-1} A(\lambda^2 I - A^2)^{-1} f + (2/(\lambda + \mu)) A(\lambda I - A)^{-1} (\mu I - A)^{-1} f \\ &= (1/(\lambda + \mu)) A_1(\lambda) f - (1/(\lambda - \mu)) A_2(\lambda) f \\ &\quad - (2/(\lambda + \mu)) A((\lambda I + A)(\lambda + \mu) + (\mu I + A)(\lambda + \mu) \\ &\quad - (\lambda I + A)(\mu I + A))(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f\end{aligned}$$

(*equation continued on p.* 704)

$$= (1/(\lambda + \mu))\, A_1(\lambda) f - (1/(\lambda - \mu))\, A_2(\lambda) f - (2/(\lambda + \mu))\, A((\lambda + \mu)^2 - \lambda\mu + (\lambda + \mu) A - A^2)(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f$$

$$= (1/(\lambda + \mu))\, A((\mu^2 I - A^2) + (\lambda^2 I - A^2) - 2(\lambda + \mu)^2 + 2\lambda\mu - 2(\lambda + \mu) A + 2A^2)(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f - (1/(\lambda - \mu))\, A_2(\lambda) f$$

$$= (1/(\lambda + \mu))\, A(-(\lambda + \mu)^2 - 2(\lambda + \mu) A)(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f - (1/(\lambda - \mu))\, A_2(\lambda) f$$

$$= A(-(\lambda + \mu) - 2A)(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f + A(\lambda + \mu)(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f$$

$$= -2A^2(\lambda^2 I - A^2)^{-1} (\mu^2 I - A^2)^{-1} f$$

$$= -2A(\lambda^2 I - A^2)^{-1} A(\mu^2 I - A^2)^{-1} f.$$

20^0 If $f \in X$, $\lambda > \omega$, $\mu > \omega$, $\lambda \neq \mu$, then

(i) $\int_0^\infty e^{-\mu s} U(t + s) f\, ds$ is continuous in $t \in R^+$,

(ii) $\| \int_0^\infty e^{-\mu s} U(t + s) f\, ds \| \leqslant (M/(\mu - \omega)) \exp(\omega t) \| f \|$,

(iii) $$\int_0^\infty e^{-\lambda t} \left(\int_0^\infty e^{-\mu s} U(t + s) f\, ds \right) dt = (1/(\lambda - \mu))\, (A(\lambda^2 I - A^2)^{-1} f - A(\mu^2 I - A^2)^{-1} f).$$

PROOF: Since $\| U(p) \| \leqslant M \exp(\omega p)$, the integral in (i) exists and (ii) follows. The continuity in (i) is established by observing that a change of variables gives.

$$\int_0^\infty e^{-\mu s} U(t + s) f\, ds = e^{\mu t} \int_t^\infty e^{-\mu p} U(p) f\, dp = e^{\mu t} \left(\int_0^\infty e^{-\mu p} U(p) f\, dp - \int_0^t e^{-\mu p} U(p) f\, dp \right).$$

Further (iii) follows, since

$$\int_0^\infty e^{-\lambda t} \left(\int_0^\infty e^{-\mu s} U(t + s) f\, ds \right) dt = \int_0^\infty e^{-(\lambda - \mu) t} dt \int_0^\infty e^{-\mu p} U(p) f\, dt - \int_0^\infty e^{-(\lambda - \mu) t} \left(\int_0^t e^{-\mu p} U(p) f\, dp \right) dt$$

(equation continued on p. 705)

$$= (1/(\lambda - \mu))\, A(\mu^2 I - A^2)^{-1} f - \int_0^\infty e^{-\mu p} U(p) f\, dp \int_p^\infty e^{-(\lambda-\mu)t}\, dt$$

$$= (1/(\lambda - \mu)\, A(\mu^2 I - A^2)^{-1} f - (1/(\lambda - \mu)) \int_0^\infty e^{-\lambda p} U(p) f\, dp$$

$$= (1/(\lambda - \mu))\, (A(\mu^2 I - A^2)^{-1} f - A(\lambda^2 I - A^2)^{-1} f).$$

21⁰ If $f \in X$, $\lambda > \omega$, $\mu > \omega$, then

(i) $\int_t^\infty e^{-\mu s} U(s + t) f\, ds$ is continuous in $t \in R^+$,

(ii) $\| \int_t^\infty e^{-\mu s} U(s - t) f\, ds \| \leqslant (M/(\mu - \omega)) \exp(-\mu t) \| f \|$,

(iii) $\int_0^\infty e^{-\lambda t} (\int_t^\infty e^{-\mu s} U(s - t) f\, ds)\, dt$

$$= (1/(\lambda + \mu))\, A(\mu^2 I - A^2)^{-1} f.$$

PROOF: We proceed as in 20⁰. (i) and (ii) follows similarly since by a change of variable

$$\int_t^\infty e^{-\mu s} U(s - t) f\, ds = e^{-\mu t} \int_0^\infty e^{-\mu p} U(p) f\, dp.$$

Now (iii) follows, since

$$\int_0^\infty e^{-\lambda t} (\int_t^\infty e^{-\mu s} U(s - t) f\, ds)\, dt$$

$$= \int_0^\infty e^{-(\lambda+\mu)t} dt \int_0^\infty e^{-\mu p} U(p) f\, dp$$

$$= (1/(\lambda + \mu))\, A(\mu^2 I - A^2)^{-1} f.$$

22⁰ If $f \in X$, $\lambda > \omega$, $\mu > \omega$, then

(i) $\int_0^t e^{-\mu s} U(t - s) f\, ds$ is continuous in $t \in R^+$,

(ii) $\| \int_0^t e^{-\mu s} U(t - s) f\, ds \| \leqslant (M/(\mu + \omega)) \exp(\omega t) \| f \|$,

(iii) $\int_0^\infty e^{-\lambda t} (\int_0^\infty e^{-\mu s} U(t - s) f\, ds)\, dt$

$$= (1/(\lambda + \mu)\, A(\lambda^2 I - A^2)^{-1} f.$$

PROOF: We proceed as in 20⁰. We easily obtain (i) and (ii), because a change of variable gives

$$\int_0^t e^{-\mu s}U(t-s)f\,ds = e^{-\mu t}\int_0^t e^{\mu p}U(p)f\,dp.$$

Further (iii) follows as

$$\int_0^\infty e^{-\lambda t}\Big(\int_0^t e^{-\mu s}U(t-s)f\,ds\Big)\,dt$$

$$= \int_0^\infty e^{-(\lambda+\mu)t}\Big(\int_0^t e^{\mu p}U(p)f\,dp\Big)\,dt$$

$$= \int_0^\infty e^{\mu p}U(p)f\,dp\Big(\int_p^\infty e^{-(\lambda+\mu)t}\,dt\Big)$$

$$= (\lambda+\mu)^{-1}\int_0^\infty e^{\mu p}e^{-(\lambda+\mu)p}U(p)f\,dp$$

$$= (\lambda+\mu)^{-1}\int_0^\infty e^{-\lambda p}U(p)f\,dp$$

$$= (\lambda+\mu)^{-1}A(\lambda^2 I - A^2)^{-1}f.$$

23^0 If $f \in D(A^4)$, $\lambda > \omega$, $\mu > \omega$, then

$$\int_0^\infty e^{-(\lambda+\mu)t}U(t)\,T(t)\,f\,dt = (\lambda+\mu)^{-1}A(\lambda I + \mu I - 2A)^{-1}f.$$

PROOF: We know that

$$d(-R(\lambda+\mu, A)\,e^{-(\lambda+\mu)t}T(t)\,f/dt)$$
$$= -R(\lambda+\mu, A)\,(-(\lambda+\mu)\,e^{-(\lambda+\mu)t}T(t)\,f + e^{-(\lambda+\mu)t}AT(t)\,f)$$
$$= R(\lambda+\mu, A)\,(\lambda I + \mu I - A)\,e^{-(\lambda+\mu)t}T(t)\,f$$
$$= e^{-(\lambda+\mu)t}T(t)\,f.$$

Therefore, using 13^0 and 16^0, we get

$$\int_0^\infty e^{-(\lambda+\mu)t}U(t)\,T(t)\,f\,dt$$

$$= -U(t)\,R(\lambda+\mu, A)\,e^{-(\lambda+\mu)t}T(t)\,f\,\Big|_0^\infty$$

$$+\int_0^\infty R(\lambda+\mu, A)\,e^{-(\lambda+\mu)t}(dU(t)/dt)\,T(t)\,f\,dt$$

$$= -R^2(\lambda+\mu, A)\,e^{-(\lambda+\mu)t}T(t)\,(dU(t)/dt)\,f\,\Big|_0^\infty$$

$$+R^2(\lambda+\mu, A)\int_0^\infty e^{-(\lambda+\mu)t}T(t)\,U(t)\,A^2 f\,dt.$$

Hence

$$(I - R^2(\lambda + \mu, A)\, A^2) \int_0^\infty e^{-(\lambda+\mu)t} U(t)\, T(t)\, f\, dt$$

$$= R^2(\lambda + \mu, A)\, Af,$$

or

$$((\lambda I + \mu I - A)^2 - A^2) \int_0^\infty e^{-(\lambda+\mu)t} U(t)\, T(t)\, f\, dt = Af,$$

or

$$(\lambda I + \mu I - 2A) \int_0^\infty e^{-(\lambda+\mu)t} U(t)\, T(t)\, f\, dt = (1/(\lambda + \mu))\, Af,$$

or

$$\int_0^\infty e^{-(\lambda+\mu)t} U(t)\, T(t)\, f\, dt = (1/(\lambda + \mu))\, (\lambda I + \mu I - 2A)^{-1}\, Af$$

$$= (1/(\lambda + \mu))\, A(\lambda I + \mu I - 2A)^{-1} f.$$

24^0 If $f \in X$, $\lambda > \omega$, $\mu > \omega$, then

$$\int_0^\infty e^{-(\lambda+\mu)t} U(t)\, T(t)\, f\, dt = (1/(\lambda + \mu))\, A(\lambda + \mu - 2A)^{-1} f.$$

PROOF: We know that the operators on both the sides of the above equality are bounded linear operators, and by 23^0, they agree on a dense domain. Hence they are equal everywhere.

25^0 If $f \in X$, $\lambda > \omega$, $\mu > \omega$, then

(i) $\int_t^\infty e^{-\mu s} T(t)\, U(s)\, f\, ds$ is continuous in $t \in R^+$,

(ii) $\| \int_t^\infty e^{-\mu s} T(t)\, U(s)\, f\, ds \| \leqslant (M^2/(\mu - \omega)) \exp(-(\mu - 2\omega)\, t) \,.\, \| f \|$,

(iii) $\int_0^\infty e^{-\lambda t} (\int_t^\infty e^{-\mu s} T(t)\, U(s)\, f\, ds)\, dt$

$$= R(\lambda, A)\, A(\mu^2 I - A^2)^{-1} f - (1/(\lambda + \mu))\, A(\lambda I + \mu I - 2A)^{-1}\, R(\lambda, A)\, f.$$

PROOF: We proceed as in 20^0 and (i), (ii) can be easily seen. As for (iii),

$$\int_0^\infty e^{-\lambda t}T(t)\,(\int_t^\infty e^{-\mu s}U(s)\,f\,ds)\,dt$$

$$= \int_0^\infty e^{-\lambda t}T(t)\,(\int_0^\infty e^{-\mu s}U(s)\,f\,ds - \int_0^t e^{-\mu s}U(s)\,f\,ds)\,dt$$

$$= \int_0^\infty e^{-\lambda t}T(t)\,A(\mu^2 I - A^2)^{-1}f\,dt$$

$$- \int_0^\infty e^{-\lambda t}T(t)\,(\int_0^t e^{-\mu s}U(s)\,f\,ds)\,dt$$

$$= R(\lambda, A)\,A(\mu^2 I - A^2)^{-1}f - \int_0^\infty e^{-\mu s}U(s)\,(\int_s^\infty e^{-\lambda t}T(t)\,f\,dt)\,ds$$

$$= R(\lambda, A)\,A(\mu^2 I - A^2)^{-1}f - \int_0^\infty e^{-(\lambda+\mu)s}U(s)\,T(s)\,R(\lambda, A)\,f\,ds$$

$$= R(\lambda, A)\,A(\mu^2 I - A^2)^{-1}f - (1/(\lambda+\mu))\,A(\lambda I + \mu I - 2A)^{-1}\,R(\lambda, A)\,f,$$

by using 24^0.

26^0 If $f \in X$, $\lambda > \omega$, $\mu > \omega$, then

(i) $\int_0^t e^{-\mu s}T(s)\,U(t)\,f\,ds$ is continuous in $t \in R^+$,

(ii) $\| \int_0^t e^{-\mu s}T(s)\,U(s)\,f\,ds \| \leqslant (M^2/(\mu - \omega))\exp(\omega t)$,

$$(1 - \exp(-(\mu - \omega)\,t))\,.\,\|f\|,$$

(iii) $\int_0^\infty e^{-\lambda t}U(t)\,(\int_0^\infty e^{-\mu s}T(s)\,f\,ds)\,dt$

$$= R(\mu, A)\,A(\lambda^2 I - A^2)^{-1}f - (1/(\lambda+\mu))\,A(\lambda I + \mu I - 2A)^{-1}\,R(\mu, A)\,f.$$

PROOF: We proceed as in 20^0 and (i), (ii) can be easily seen. As for (iii),

$$\int_0^\infty e^{-\lambda t}U(t)\,(\int_0^t e^{-\mu s}T(s)\,f\,ds)\,dt$$

$$= \int_0^\infty e^{-\mu s}T(s)\,(\int_s^\infty e^{-\lambda t}U(t)\,f\,dt)\,ds$$

$$= R(\mu, A)\,A(\lambda^2 I - A^2)^{-1}f - (1/(\lambda+\mu))\,A(\lambda I + \mu I - 2A)^{-1}\,R(\mu, A)\,f,$$

as in (iii) of 25^0.

27^0 Let $f \in X$ be fixed, and let us define an auxiliary function $\mathbf{\Lambda}(t, s)$ for t, $s \in R^+$ with values in X, as follows :

(i) for $t > s$, $\mathbf{\Lambda}(t, s) = U(t + s) + U(t - s) - 2T(s)\, U(t)$,

(ii) for $t < s$, $\mathbf{\Lambda}(t, s) = U(t + s) + U(s - t) - 2T(t)\, U(s)$,

(iii) for $t = s$, $\mathbf{\Lambda}(t, s) = U(2t) - 2U(t)\, T(t)$.

Now we establish some properties of $\mathbf{\Lambda}$:

(a_1) $\mathbf{\Lambda}(t, s) = \mathbf{\Lambda}(s, t)$, for every pair $t, s \in R^+$,

(a_2) for each fixed $t \in R^+$, the function $\mathbf{\Lambda}(t, .)$ is continuous on R^+, and

(a_3) $\| \mathbf{\Lambda}(t, s) \| \leqslant 4M^2 \exp(\omega t) \exp(\omega s)$, for every pair $t, s \in R^+$.

Proof : (a_1) is obvious, (a_2) is a consequence of 16^0 and 17^0, and (a_3) of 16^0.

28^0 Now let us define for every $\mu > \omega$ and $t \in R^+$:

$$\mathbf{\Lambda}_1(t, \mu) = \int_0^\infty e^{-\mu s} \mathbf{\Lambda}(t, s)\, ds.$$

This definition is meaningful by 27^0 (a_2) and (a_3). We establish some properties of $\mathbf{\Lambda}_1$:

(b_1) for every fixed $\mu > \omega$, the function $\mathbf{\Lambda}_1(., \mu)$ is continuous on R^+,

(b_2) for every $t \in R^+$ and $\mu > \omega$, we have

$$\| \mathbf{\Lambda}_1(t, \mu) \| \leqslant (4M^2/(\mu - \omega)) \exp(\omega t).$$

Proof : (b_1) is an easy consequence of 27^0 (a_1) – (a_2), and (b_2) of 27^0 (a_3).

29^0 Now let us define for every $\lambda > \omega$, $\mu > \omega$:

$$\mathbf{\Lambda}_2(\lambda, \mu) = \int_0^\infty e^{-\lambda t} \mathbf{\Lambda}_1(t, \mu)\, dt.$$

This definition is meaningful by 28^0 (b_1) – (b_2). Now by the use of 20^0 to 28^0, we obtain

$$\mathbf{\Lambda}_2(\lambda, \mu) = 2A(\lambda^2 I - A^2)^{-1} A(\mu^2 I - A^2)^{-1},$$

for every $\lambda > \omega$, $\mu > \omega$.

30^0 $\{U(t)\}$ is a modified sine operator.

Proof : By 18^0 and Theorems 16a and 16b, Chapter VII, Widder (1966), for every $t \in R^+$ and $\mu > \omega$, we have

$$\mathbf{\Lambda}_1(t, \mu) = -\, 2U(t)\, A(\mu^2 I - A^2)^{-1}.$$

Repeating this argument, we obtain

$$\Lambda(t, s) = -2U(t)\, U(s),$$

for every pair $t, s \in R^+$.

Hence, for $t > s$, we obtain

$$U(t+s) + U(t-s) - 2T(s)\, U(t) = -2U(t)\, U(s),$$

or

$$U(t+s) + U(t-s) + 2U(t)\, U(s) = 2T(s)\, U(t).$$

This completes the proof.

31⁰ U is a regular modified sine operator.

PROOF: Consequence of 30⁰ and 17⁰.

32⁰ If $f \in D(A^4)$ and $t \in R^+$, then

$$U(t)\, f = t\, Af + \int_0^t \left(\int_0^s U(p)\, A^2 f\, dp \right) ds.$$

PROOF: The existence of the integral on the right side follows from 16⁰. It suffices, therefore, only to prove that, for $l \in X^*$, one has

$$l(U(t)\, f) = tl(Af) + \int_0^t \left(\int_0^s l(U(p)\, A^2 f)\, dp \right) ds,\ f \in D(A^4).$$

Using successively 16⁰, 10⁰ and 3⁰, we obtain

$$l(U(t)\, f) = \Psi(f, l)\,(t) = tl(Af) + \int_0^t \left(\int_0^s \Phi(A^2 f, l)\,(p)\, dp \right) ds.$$

Now, since $f \in D(A^4)$, we have $A^2 f \in D(A^2)$, and hence, by 5⁰, we obtain

$$l(U(t)\, f) = tl(Af) + \int_0^t \left(\int_0^s \Psi(A^2 f, l)\,(p)\, dp \right) ds.$$

Now by 10⁰ and 16⁰,

$$l(U(t)\, f) = tl(Af) + \int_0^t \left(\int_0^s l(U(p)\, A^2 f)\, dp \right) ds,\ f \in D(A^4).$$

33⁰ If $f \in D(A^4)$, then $f \in D(E)$ and $f \in D(F)$, and $Ef = Af$, $Ff = A^2 f$.

PROOF: Obviously it will be sufficient to prove that for every $f \in D(A^4)$, we have

$$(U(t)f/(t) \to Af, \quad \text{as} \quad t \to 0, \ t \in R^+,$$

and $$(2(T(t)f - U(t)f - f)/t^2) \to A^2 f, \quad \text{as} \quad t \to 0, \ t \in R^+.$$

By 32⁰, we obtain

$$(U(t)f/t) = Af + (1/t)\int_0 \Big(\int_0^s U(p)\, A^2 f\, dp\Big)\, ds, \ f \in D(A^4).$$

Therefore,

$$\| (U(t)f/t) - Af \| \leqslant (t/2) \sup_{0 \leqslant p \leqslant t} \| U(p)\, A^2 f \|.$$

Using 17ᵉ, we have

$$\lim_{\substack{t \to 0, \\ t > 0}} \| (U(t)f/t) - Af \| = 0,$$

that is, $(U(t)f/t) \to Af$, as $t \to 0$, $t > 0$, for $f \in D(A^4)$. Also, for $f \in D(A^4)$,

$$\begin{aligned} &(2/t^2)\,(T(t)f - U(t)f - f) - A^2 f \\ &\quad = ((2/t^2)\,(T(t)f - tAf - f) - A^2 f) \\ &\quad\quad - (2/t^2)\int_0^t \Big(\int_0^s U(p)\, A^2 f\, dp\Big)\, ds. \end{aligned}$$

Hence

$$\begin{aligned} &\| (2/t^2)\,(T(t)f - U(t)f - f) - A^2 f \| \\ &\quad \leqslant \| (2/t^2)\,(T(t)f - tAf - f) - A^2 f \| + \sup_{0 \leqslant p \leqslant t} \| U(p)\, A^2 f \|, \end{aligned}$$

which tends to zero as $t \to 0$, $t > 0$.

Hence

$$(2/t^2)\,(T(t)f - U(t)f - f) \to A^2 f, \quad \text{as} \quad t \to 0, \ t > 0,$$

for $f \in D(A^4)$.

34⁰ We claim that $\bar{E} = A$ and $\bar{F} = A^2$.

PROOF : For $f \in D(A^4)$,

$$(\lambda^2 I - \bar{F})\,(\lambda^2 I - A^2)^{-1} f = f.$$

Since $\bar{F}$ is closed and $D(A^4)$ is dense in X, we obtain for every $f \in X$, a sequence $\{f_n\}$ in $D(A^4)$ such that $f_n \to f$. Also

$$(\lambda^2 I - \bar{F})\,(\lambda^2 I - A^2)^{-1} f_n = f_n.$$

Hence $\lim_{n\to\infty} (\lambda^2 I - \bar{F})(\lambda^2 I - A^2)^{-1} f_n$ exists and equal to f; that is,

$$\lim_{n\to\infty} (\lambda^2 I - \bar{F})(\lambda^2 I - A^2)^{-1} f_n = f,$$

But $(\lambda^2 I - A^2)^{-1} f_n \to (\lambda^2 I - A^2)^{-1} f$, as $n \to \infty$. Using closedness of $\bar{F}$ we get

$$(\lambda^2 I - \bar{F})(\lambda^2 I - A^2)^{-1} f = f,$$

that is, $(\lambda^2 I - \bar{F})^{-1} = (\lambda^2 I - A^2)^{-1}$, and $\bar{F} = A^2$. Also, for $f \in D(A^4)$,

$$(\lambda I - \bar{E}) R(\lambda, A) f = f.$$

Since $D(A^4)$ is dense in X, for every $f \in X$, we can find a sequence $\{f_n\}$, $f_n \in D(A^4)$ such that $f_n \to f$.

Also $(\lambda I - \bar{E}) R(\lambda, A) f_n = f_n$.

Hence $\lim_{n\to\infty} (\lambda I - \bar{E}) R(\lambda, A) f_n$ exists,

and $\lim_{n\to\infty} (\lambda I - \bar{E}) R(\lambda, A) f_n = f.$

Since $\lim_{n\to\infty} R(\lambda, A) f_n = R(\lambda, A) f$ and $\bar{E}$ is a closed operator, we have

$$(\lambda I - \bar{E}) R(\lambda, A) f = f,$$

that is, $(\lambda I - \bar{E})^{-1} = R(\lambda, A) = (\lambda I - A)^{-1}$, thus proving that $\bar{E} = A$.

35^0 Proof of the Theorem is a consequence of 1^0 – 34^0.

3. Examples

Example 1 — In the example 1 of Ramesh Chander and Buche (1981b), $\{U(t), t \in R^+\}$, is defined on $C(R)$ as

$$[U(t) f](x) = (\tfrac{1}{2})(f(x + at) - f(x - at)),$$

where $a \neq 0$ is some constant, $f \in C(R)$ and $x \in R$, the real line. It has been proved (Chander and Buche 1981b) that $\{U(t)\}$ is a regular modified sine operator, and $\| U(t) \| \leqslant 1$.

Now we shall find the Laplace transform of this modified sine operator and the approximating sequence given by $[V_n(n/t) f](x)$ as discussed in Proposition 3. Let us write for $\lambda > 0$,

$$[L(\lambda) f](x) = \int_0^\infty (\exp(-\lambda p))\, U(p) f(x)\, dp$$

$$= (1/2) \int_0^\infty (\exp(-\lambda p)) f(x + ap)\, dp$$

$$- (1/2) \int_0^\infty (\exp(-\lambda p)) f(x - ap)\, dp$$

$$= (1/2a) \int_x^\infty (\exp(-\lambda(p - x)/a)) f(p)\, dp$$

$$- (1/2a) \int_x^\infty (\exp(-\lambda(x - p)/a) f(p)\, dp.$$

which is the Laplace transform of $\{U(t)\}$. Differentiating $L(\lambda)$ n times, we get

$$(d^n L(\lambda) f(x)/d\lambda^n) = ((-1)^n/2) \Big(\int_x^\infty (\exp(-\lambda(p - x)/a))\, (p - x)^n/a^{n+1}) f(p)\, dp$$

$$- \int_x^\infty (\exp(-\lambda(x - p)/a))\, ((x - p)^n/a^{n+1}) f(p)\, dp\Big).$$

So the approximating sequence for $[U(t) f](x)$ is given by

$$[V_n(n/t) f](x) = ((n/t)^{n+1}/2(n!)) \Big(\int_x^\infty (\exp(-n(p - x)/ta))\, ((p - x)^n/a^{n+1}) f(p)\, dp$$

$$- \int_0^\infty (\exp(-n(x - p)/ta))\, ((x - p)^n/a^{n+1}) f(p)\, dp\Big),$$

$n = 0, 1, 2, \ldots, -\infty < x < \infty$.

Example 2 — In the Example 2 of Ramesh Chander and Buche (1981b), $\{U(t);\ t \in R^+\}$ is defined on $C(R)$ as

$$[U(t) f](x) = \sum_{k=0}^{\infty} ((\alpha t)^{2k+1}/(2k + 1)!) f(x - (2k + 1)\, \mu),$$

where $\mu > 0$ is some constant, (cf. Buche 1971). It has been proved that $\{U(t)\}$ is a regular modified sine operator with $\| U(t) \| \leqslant \exp(\alpha t)$.

The Laplace transform $L(\lambda)$ of $\{U(t)\}$, for $\lambda > \alpha$, is given by

$$L(\lambda) f(x) = \int_0^\infty (\exp(-\lambda t))\, [U(t) f](x)\, dt$$

$$= \int_0^\infty (\exp(-\lambda t)) \Big(\sum_{k=0}^{\infty} ((\alpha t)^{2k+1}/(2k + 1)!) f(x - (2k + 1)\, \mu)\Big)\, dt.$$

Because of the uniform convergence of the series, it is term by term integrable. Hence

$$L(\lambda) f(x) = \sum_{k=1}^{\infty} (\alpha^{2k+1}/(2k+1)!)\, f(x-(2k+1)\mu) \int_0^{\infty} (\exp(-\lambda t))\, t^{2k+1} dt$$

$$= (1/\lambda) \sum_{k=0}^{\infty} (\alpha/\lambda)^{2k+1} f(x-(2k+1)\mu), \quad \lambda > \alpha$$

which is the required Laplace transform of the above defined modified sine operator. Differentiating $L(\lambda)$ n times with respect to λ, we get

$$(d^n L(\lambda) f(x)/d\lambda^n)$$

$$= (-1)^n/\lambda^{n+1}) \sum_{k=0}^{\infty} ((2k+n+1)!/(2k+1)!)\, (\alpha/\lambda)^{2k+1} f(x-(2k+1)\mu).$$

So the approximating sequence for $\{U(t)\}$ is given by

$$[V_n(n/t) f](x)$$

$$= ((-1)^n/n!)\, (n/t)^{n+1} (d^n L(\lambda) f(x)/d\lambda^n)|_{\lambda=n/t}$$

$$= (1/n!) \sum_{k=0}^{\infty} ((2k+1+n)!/(2k+1)!)\, (\alpha t/n)^{2k+1} f(x-(2k+1)\mu)$$

$$= \sum_{k=0}^{\infty} \binom{2k+1+n}{2k+1} (\alpha t/n)^{2k+1} f(x-(2k+1)\mu) \quad \text{for} \quad n > \alpha t.$$

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