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A NOTE ON A SET-THEORETIC RESULT OF MROWKA

V. KANNAN
School of Mathematics, University of Hyderabad, Hyderabad 500001

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Here we give a considerably shorter and entirely different proof of a recent set-theoretic proposition due to Mrowka, that has been found to have several interesting consequences.

Theorem — Let X be an infinite set. Let $\mathbf{F}$ be a family of functions on X such that (1) $|\mathbf{F}| \leqslant |X|$ and (2) $|X \setminus A| = |X|$ whenever A is a subset of X such that some member of $\mathbf{F}$ is constant on A. Then there exists a permutation π of X such that $f \circ \pi$ is not in $\mathbf{F}$, whenever f is in $\mathbf{F}$.

PROOF : Let m be the cardinality of X and let ω_m be its initial ordinal. Well-order X to the type of ω_m, say as $x_1, x_2, \ldots x_\alpha \ldots$. We may assume $\mathbf{F}$ to be infinite. We fix a subset A of $[1, \omega_{|F|})$ such that both A and its complement are cofinal in $[1, \omega_{|F|})$. To be more specific, let

$$A = \begin{cases} \text{the set of odd ordinals less than } \omega_{|F|} \text{ if } \mathbf{F} \text{ is countable; the set} \\ \text{of non-limit ordinals if } \mathbf{F} \text{ is uncountable.} \end{cases}$$

We fix an arbitrary bijection b from the set of all pairs (f, g) of members of $\mathbf{F}$ to the above set A. For every $\alpha < \omega_m$, we now associate a subset of X by defining $B_\alpha = f^{-1}(g(x_\alpha))$ if $\alpha \in A$ and (f, g) is the unique pair of elements in $\mathbf{F}$ such that $b(f, g) = \alpha$, and by letting B_α to be empty if α is not in A. Note that by condition (2) the complement of each B_α has cardinality m.

Now we define π recursively on X by transfinite induction. $\pi(x_1)$ is defined as the first element of $X \setminus B_1$. If $\pi(x_\beta)$ has been already defined for every $\beta < \alpha$, then $\pi(x_\alpha)$ is defined as the first element of the complement of the set $B_\alpha \cup \{\pi(x_\beta) : \beta < \alpha\}$. (Note that this set is nonempty). Thus π is defined for all elements of X.

We claim that π has the required properties. Since, whenever $\beta < \alpha$, $\pi(x_\alpha)$ is defined to be an element different from $\pi(x_\beta)$, it is clear that π is one-to-one. Suppose $\alpha_0 < \omega_m$ is such that x_{α_0} is not in the range of π. Then π maps all elements outside A into the set $\{x_\alpha : 1 \leqslant \alpha < \alpha_0\}$. This is impossible since π is one-to-one and since this set has cardinality $< m$. This proves that π is onto.

Suppose that f and $f \circ \pi$ both belong to F. Calling $g = f \circ \pi$ and letting $\alpha = b(f, g)$, we have $\pi(x_\alpha) \notin B_\alpha = f^{-1}(g(x_\alpha))$. Therefore $f(\pi(x_\alpha)) \neq g(x_\alpha)$, a contradiction to $g = f \circ \pi$.

Remarks : This theorem has been first proved by Mrowka (1977) as a stronger form of the results obtained in two earlier papers. The proof there uses intricate arguments and set-theoretic results such as König's theorem. Our proof is straightforward, after a proper choice of ordering on suitable collections.

Reference

Mrowka, S. (1977). Some set-theoretic constructions in topology. *Fund. Math.*, XCIV, 83-92.