*Indian J. pure appl. Math.*, 12(6) : 715–721, June 1981

# ON STABILITY OF DERIVO-PERIODIC SOLUTIONS OF NON-HOLONOMIC SYSTEMS

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(*Received* 4 *February* 1980; *after revision* 17 *September* 1980)

The non-holonomic periodic perturbation of a holonomic system is considered. A general condition and an expression for stability criteria of the derivo-periodic solution are given.

In this paper we consider the perturbed system containing a small parameter while it is assumed that the unperturbed system is holonomic, has a *D*-periodic solution, the perturbation is non-holonomic and is controllably periodic. The stability criterion was studied by Farkas (1971, 1972) and El-Owaidy (1975). Here we shall report further studies in a general case. For the sake of convenience, some results of Farkas and El-Owaidy are summarized in section 1.

§1. Consider the system

$$\frac{dx}{dt} = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right) = \tilde{F}\left(\frac{t}{\tau}, x, \mu, \tau\right) \qquad \ldots(1)$$

where $x, f, g$ are $n$-dimensional vectors and $t, \mu, \tau$ are real scalars and the function on the right-hand side is analytic in the region $I_t \times \Omega \times I_\mu \times I_\tau$ where

$$I_t = \{t : -\infty < t < +\infty\},$$

$\Omega$ is an open and connected region in $n$-dimensional space of $x$,

$$I_\mu = \{\mu : | \mu | < \alpha\}, \quad \alpha > 0,$$

$$I_\tau = \{\tau : | \tau - \tau_0 | < \beta\}, \quad 0 < \beta < \tau_0.$$

Assume that the function

$$\tilde{F}\left\{\frac{t}{\tau}, x, \mu, \tau\right\}$$

is periodic in $t$ with period $\tau \in I_\tau$ and is also periodic in the vector $x$ with vector period a $\tau$.

*Definition* — We say that the continuously differentiable vector function $\varphi(t)$ is *D*-periodic with period $\tau$ if its derivative is periodic with least period (cf. Farkas 1971).

Together with (1) we consider the unperturbed holonomic system

$$\frac{dx}{dt} = f(x) \qquad ...(2)$$

It will be assumed that the system (2) has a non-constant $D$-periodic solution

$$p(t) = \gamma(t) + a^0 t \qquad (\text{such that } \gamma(t + \tau_0) = \gamma(t)),$$

with period $\tau_0 > 0$, coefficient vector $a^0$. The first variational system of (2) corresponding to $p(t)$ is

$$\frac{dy}{dt} = f'_x\,(p(t))\,y \qquad ...(3)$$

and its fundamental matrix solution is denoted by $Y(t)$. It is assumed that $n - 1$ characteristic multipliers of system (3) are in modulus less than one.

It was proved by Farkas (1971) that the system (1) has a unique $D$-periodic solution $\varphi(t, \mu, \theta)$ with period $\tau(\mu, \theta)$. The first variational system of (1) corresponding to $\varphi(t, \mu, \theta)$ is

$$\frac{dy}{dt} = \left[f'_x\,(\varphi(t, \mu, \theta)) + \mu g'_x\left(\frac{t}{\tau}, \varphi(t,\mu, \theta), \mu, \tau(\mu, \theta)\right)\right] Y. \qquad ...(4)$$

and its fundamental matrix solution is denoted by $Y(t; \mu, \theta)$, for which $Y(0, \mu, \theta) = U$ ($U$ : the unit matrix) holds, and by $C(\mu, \theta) = Y(\tau, \mu, \theta)$, the characteristic matrix of (4) for which $C(0, 0) = Y(\tau_0, 0, 0) = Y(\tau_0)$ holds. Let $\lambda(\mu, \theta)$ also denote the characteristic multiplier of $C(\mu, \theta)$ for which $\lambda(0, 0) = 1$ holds.

It was proved by Farkas (1972) that the $D$-periodic solution $\varphi(t; \mu, \theta)$ of (1) is asymptotically stable if $\mu\lambda'_\mu\,(0, 0) < 0$. Also if $\lambda'_\mu\,(0, 0) = 0$ it was proved by the author (El-Owaidy 1975) that the solution is asymptotically stable if $\lambda''_{\mu\mu}\,(0, 0) < 0$.

§2. Now we shall study the stability of the $D$-periodic solution of the system (1) in the general case:

*Theorem* 1 — Under the assumptions stated above, if 1 is a simple characteristic multiplier of (3) and the remaining $(n - 1)$ characteristic multipliers are in modulus less than 1 and

$$\left.\begin{aligned} &\lambda^{(r)}_{\mu\cdots\mu}\,(0, 0) = 0, \quad r = 1, 2, ..., k - 1 \\ &\lambda^{(k)}_{\mu\cdots\mu}\,(0, 0) < 0, \quad k \text{ even} \end{aligned}\right\} \qquad ...(5)$$

then there exists $\rho > 0$ such that for

$$0 < |\mu| < \rho, \quad \theta = 0,$$

the $D$-periodic solution $\varphi(t, \mu, 0)$ of system (1) is asymptotically stable.

PROOF: The proof is analogous to that of Theorem 1 (cf. El-Owaidy 1975) and should, therefore, be omitted here.

It should be noted that if

$$\lambda^{(r)}_{\mu\ldots\mu}(0, 0) = 0, \quad r = 1, 2, \ldots, k-1,$$

$$\lambda^{(k)}_{\mu\ldots\mu}(0, 0) > 0,$$

then the $D$-periodic solution is unstable.

Since the characteristic matrix $C(\mu, \theta)$ is an analytic function of its arguments, it can be expanded for fixed $\theta$ in powers of $\mu$, i.e.

$$C(\mu, \theta) = C_0(\theta) + \mu C_1(\theta) + \mu^2 C_2(\theta) + \ldots$$
$$\ldots + \mu^k C_k(\theta) + R(\mu, \theta). \qquad \ldots(6)$$

where $R$ is an analytic function.

Let the characteristic polynomial of $C(\mu, \theta)$ be denoted by

$$(-1)^n d(\lambda, \mu, \theta) = \det [C(\mu, \theta) - \lambda U]. \qquad \ldots(7)$$

Using Poincaré's method we can give an effective form for the stability condition $\lambda^{(k)}_{\mu\ldots\mu}(0, 0)$ of Theorem 1.

*Theorem* 2 — Under the condition of Theorem 1

$$\lambda^{(k)}_{\mu\ldots\mu}(0, 0) = \frac{(-1)^{n-1}(K!)}{d'_\lambda(1, 0, 0)} \left\{ \sum_{\substack{i_1, i_2, \ldots i_k = 1 \\ i_1 \neq i_2 \neq \ldots \neq i_k}}^{n} \det M_{(i_1, i_2, \ldots i_k)} \right.$$
$$+ \sum_{\substack{i_1, i_2, \ldots i_{k-1} = 1 \\ i_1 \neq i_2 \neq \ldots \neq i_{k-1}}}^{n} \det M_{(i_1, i_2, i_{k-1})} + \sum_{\substack{i_1, i_2 \ldots i_j = 1 \\ i_1 \neq i_2 \neq \ldots \neq i_j}}^{n} \det (M_{i_1, i_2, \ldots i_j})$$
$$+ \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} \det M_{(i_1, i_2)} + \sum_{i_1 = 1}^{n} \det M_{(i_1)} \qquad \ldots(8)$$

where

$$M_{(i_1,i_2,\cdots i_k)} = \begin{vmatrix} c_1^0 - u_1 \\ \cdots \quad \cdots \quad \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^1 \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_2-1}^0 - u_{i_2-1} \\ c_{i_2}^1 \\ c_{i_2+1}^0 - u_{i_2+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_k-1}^0 - u_{i_k-1} \\ c_{i_k}^1 \\ c_{i_k+1}^0 - u_{i_k+1} \\ \cdots \quad \cdots \quad \cdots \\ c_n^0 - u_n \end{vmatrix}, \quad M_{(i_1,i_2,\cdots i_{k-1})} = \begin{vmatrix} c_1^0 - u_1 \\ \cdots \quad \cdots \quad \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^1 \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_2-1}^0 - u_{i_2-1} \\ c_{i_2}^1 \\ c_{i_2+1}^0 - u_{i_2+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_{k-1}-1}^0 - u_{i_{k-1}-1} \\ c_{i_{k-1}}^2 \\ c_{i_{k-1}+1}^0 - u_{i_{k-1}+1} \\ \cdots \quad \cdots \quad \cdots \\ c_n^0 - u_n \end{vmatrix}, \ \ldots, \ M_{(i_1,i_2,\cdots i_j)} = \begin{vmatrix} c_1^0 - u_1 \\ \cdots \quad \cdots \quad \cdots \\ c_{i_1-1}^0 - u_{i-1} \\ c_{i_1}^{r_1} \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_2-1}^0 - u_{i_2-1} \\ c_{i_2}^{r_2} \\ c_{i_2+1}^0 - u_{i_2+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_j-1}^0 - u_{i_j-1} \\ c_{i_j}^{k-\sum_{s=1}^{j-1} r_s} \\ c_{i_j+1}^0 - u_{i_j+1} \\ \cdots \quad \cdots \quad \cdots \\ c_n^0 - u_n \end{vmatrix}$$

$$M_{(i_1,i_2)} = \begin{vmatrix} c_1^0 - u_1 \\ \cdots \quad \cdots \quad \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^1 \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \quad \cdots \quad \cdots \\ c_{i_2-1}^0 - u_{i_2-1} \\ c_{i_2}^{k-1} \\ c_{i_2+1}^0 - u_{i_2+1} \\ \cdots \quad \cdots \quad \cdots \\ c_n^0 - u_n \end{vmatrix} \qquad M_{(i_1)} = \begin{vmatrix} c_1^0 - u_1 \\ c_2^0 - u_2 \\ \cdots \quad \cdots \quad \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_i^k \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \\ c_n^0 - u_n \end{vmatrix} \quad \ldots(9)$$

and $c_j^0, c_j^1, c_j^2, \dots c_j^K$ and $u_j$ are the $j$th row vectors of the matrices:

$C_0(0), C_1(0), C_2(0), \dots, C_K(0)$ and $U$, respectively, $r_s \geqslant 1$, $K - \sum_{s=1}^{j-1} r_s > 0$.

PROOF: Splitting up the determinant given by (7) by terms in its last row vector using the expansion (6) to a sum of $(K+1)$ determinants we obtain

$$(-1)^n d(\lambda, u, \theta) = \det \begin{vmatrix} c_1^0 - \lambda u_1 + \mu c_1^1 + \mu^2 c_1^2 + \dots + \mu^k c_1^k + \mu^{k+1} r_1 \\ c_2^0 - \lambda u_2 + \mu c_2^1 + \mu^2 c_2^2 + \dots + \mu^k c_2^k + \mu^{k+1} r_2 \\ \dots \\ c_{n-1}^0 - \lambda u_{n-1} + \mu c_{n-1}^1 + \mu^2 c_{n-1}^2 + \dots + \mu^k c_{n-1}^k + \mu^{k+1} r_{n-1} \\ c_n^0 - \lambda u_n \end{vmatrix}$$

$$+ \mu \det \begin{vmatrix} c_1^0 - \lambda u_1 + \mu c_1^1 + \mu^2 c_1^2 + \dots + \mu^k c_1^k + \mu^{k+1} r_1 \\ c_2^0 - \lambda u_2 + \mu c_2^1 + \mu^2 c_2^2 + \dots + \mu^k c_2^k + \mu^{k+1} r_2 \\ \dots \\ c_{n-1}^0 - \lambda u_{n-1} + \mu c_{n-1}^1 + \mu^2 c_{n-1} + \dots + \mu^k c_{n-1}^k + \mu^{k+1} r_{n-1} \\ c_n^1 \end{vmatrix} + \dots$$

$$+ \mu^K \det \begin{vmatrix} c_1^0 - \lambda u_1 + \mu c_1^1 + \mu^2 c_1^2 + \dots + \mu^k c_1^k + \mu^{k+1} r_1 \\ c_2^0 - \lambda u_2 + \mu c_2^1 + \mu^2 c_2^2 + \dots + \mu^k c_2^k + \mu^{k+1} r_2 \\ \dots \\ c_{n-1}^0 - \lambda u_{n-1} + \mu c_{n-1}^1 + \mu^2 c_{n-1}^2 + \dots + \mu^k c_{n-1}^k + \mu^{k+1} r_{n-1} \\ c_n^k \end{vmatrix} + \mu^{k+1} \det \begin{vmatrix} r_1 \\ r_2 \\ r_3 \\ \dots \\ \dots \\ \dots \\ r_{n-1} \\ r_n \end{vmatrix}$$

Also splitting up the resulting determinants by terms in the $(n-1)$th, $(n-2)$th, ... and the first rows, and by putting $\lambda = 1$, $\theta = 0$ in the resulting determinants we obtain

$$(-1)^n d(1, \mu, 0) = \det [C_0(0) - U] + \mu N_1 + \mu^2 N_2 + \dots + \mu^k N_k + \mu^{k+1} R_{K+1}(u, 0) \qquad \dots(10)$$

where

$$N_1 = \sum_{i_1=1}^{n} \det \begin{vmatrix} c_1^0 - u_1 \\ c_2^0 - u_2 \\ \cdots \cdots \cdots \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^1 \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ c_n^0 - u_n \end{vmatrix}, \quad N_2 = \sum_{\substack{i_1, i_1=1 \\ i_1 \neq i_2}}^{n} \begin{vmatrix} c_1^0 - u_1 \\ c_2^0 - u_2 \\ \cdots \cdots \cdots \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^1 \\ c_{i_1+1}^0 + u_{i_1+1} \\ \cdots \cdots \cdots \\ c_{i_2-1}^0 - u_{i_2-1} \\ c_{i_2}^1 \\ c_{i_2+1}^0 - u_{i_2+2} \\ \cdots \cdots \cdots \\ c_n^0 - u_n \end{vmatrix} + \sum_{i_1=1}^{n} \det \begin{vmatrix} c_1^0 - u_1 \\ c_2^0 - u_2 \\ \cdots \cdots \cdots \cdots \\ c_{i_1-1}^0 - u_{i_1-1} \\ c_{i_1}^2 \\ c_{i_1+1}^0 - u_{i_1+1} \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ \cdots \cdots \cdots \\ c_n^0 - u_n \end{vmatrix}$$

$$N_k = \sum_{\substack{i_1, i_2, \ldots i_k=1 \\ i_1 \neq i_2 \neq \ldots \neq i_k}}^{n} \det M_{(i_1, i_2, \cdots i_k)} + \ldots + \sum_{\substack{i_1, i_2=1 \\ i_1 \neq i_2}}^{n} \det M_{(i_1, i_2)}$$

$$+ \sum_{i_1=1}^{n} \det M_{(i_1)}$$

and $M_{(i_1, i_2, \cdots, i_k)}, \ldots, M_{(i_1, i_2)}, M_{(i_1)}$ are defined by (9).

Since 1 is a simple root of the polynomial $d(1; 0, 0)$, i.e. $d(1, 0, 0) = 1$, $d'_\lambda\,(1, 0, 0) \neq 0$ and since $\lambda(\mu, 0)$ is an analytic function in a neighbourhood of $(0, 0)$ such that $\lambda(0, 0) = 1$, hence $\lambda(\mu, 0)$ can be expressed from the equation $d(\lambda; \mu, 0)$ as

$$\lambda(\mu, 0) = 1 + \mu\lambda'_\mu\,(0, 0) + \frac{\mu^2}{2!}\,\lambda''_{\mu\mu}\,(0, 0) + \ldots + \frac{\mu^k}{K!}\,\lambda^{(K)}_{\mu\cdots\mu}\,(0,0) + O\,(\mu^k). \qquad \ldots(11)$$

$\lambda^{(K)}_{\mu\cdots\mu}\,(0, 0)$ can be obtained from the relation $d(\lambda; \mu, 0)$ by differentiating $K$ times with espect to $\mu$ and taking into consideration $\lambda^{(r)}_{\mu\cdots\mu}\,(0, 0) = 0$, $r = 1, 2, \ldots, K-1$, and by putting $\lambda = 1$, $\mu = 0$ we obtain

$$\lambda^{(K)}_{\mu\cdots\mu}\,(0, 0) = -\,d^{(K)}_{\mu\mu-\mu}\,(1, 0, 0)/d'_\lambda\,(1; 0, 0). \qquad \ldots(12)$$

If we differentiate (10) $K$ times with respect to $\mu$, put $\mu = 0$, using (12) we obtain (8); this completes the proof.

To use the expression (8) we need to calculate the matrices $C^0(0)$, $C^1(0)$, ..., $C^K(0)$, which can be obtained in the same manner as for $C^1(0)$ (cf. Farkas 1972) and $C^2(0)$ (cf. El-Owaidy 1975), since the computations for these matrices are in general very long and tedious.

*Notes*:

(1) It is clear that the sum of the upper indices of all the row vectors must be $K$ in all the above determinants.

(2) The results obtained here are closely related to those of Loud (cf. Loud 1959) though we obtained a general expression for stability criteria and the problem and approach are different.

(3) We studied here the case when the parameter $\theta = 0$. If $\theta \neq 0$ then many critical cases arise (cf. Farkas 1973).

(4) If the periodic solution of (3) is not isolated [i.e. 1 is not a simple characteristic multiplier of (3)] the above stability criteria does not hold and needs more investigation.

## Acknowledgement

The author wishes to express his appreciation to Professor Miklós Farkas, Head of the Mathematical Department, Budapest Technical University, for useful discussion and comments. He would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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