Indian J. pure appl. Math., **12**(6) : 722–727, June 1981

LINEAR FUNCTIONALS ON THE SPACES OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES OVER NON-ARCHIMEDIAN FIELDS

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(*Received* 14 *April* 1980; *after revision* 19 *December* 1980)

Continuous scalar homomorphisms for the spaces of entire functions of several variables over a complete non-archimedian field K are characterised for pointwise multiplication and Hadamard composition of entire functions. If B is the class of bounded linear functionals on the space of entire functions of several variables, B is proved to be a non-archimedian Banach space. It is noted that B and $K \times K \times K$ are isometrically isomorphic.

1. INTRODUCTION

Recently Jain and Jain (1977) have considered linear spaces of entire functions of several variables over non-archimedian field K endowed with a complete locally K-convex topology. The object of the present paper is to characterise the classes of scalar homomorphisms and metrically bounded linear functionals on $K \langle x_1, x_2 \rangle$.

Let K be a non-archimedian field complete under the metric of valuation. Let $K \langle x_1, x_2 \rangle$ denote the set of all functions $f : K \times K \rightarrow K$ such that

$$f(x_1, x_2) = \sum_{m+n=0}^{\infty} a_{mn} x_1^m x_2^n \qquad \ldots(2.1)$$

where $\quad a_{mn}$ and $x_1, x_2 \in K$ and $\lim\limits_{m+n \rightarrow \infty} |a_{mn}|^{1/(m+n)} = 0. \qquad \ldots(2.2)$

From (1977), it is known that $K \langle x_1, x_2 \rangle$ is a complete metric space, if the f-norm on $K \langle x_1, x_2 \rangle$ is defined as

$$f = \sup \{ |a_{00}| , |a_{mn}|^{1/(m+n)} \; m, n \geqslant 0, m + n \geqslant 1\}$$

The following theorem noted in (1977) characterises the continuous linear functionals on $K \langle x_1, x_2 \rangle$

Theorem A — Any continuous linear functional F on $K \langle x_1, x_2 \rangle$ is of the form

$$F(f) = \sum_{m, n=0}^{\infty} a_{mn} c_{mn}, \; f(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n \qquad \ldots(2.3)$$

where $\{ | c_{mn} |^{1/(m+n)} \}$ is bounded. Conversely if $\{ | c_{mn} |^{1/(m+n)} \}$ is bounded, then F defined by (2.3) is a continuous linear functional.

2. The Class of Continuous Scalar Homomorphisms

Following Wilansky (1964) a scalar homomorphisms is defined as follows.

Definition 1 — Let $f, g \in K \langle x_1, x_2 \rangle$ and $\alpha \in K$. A scalar homomorphism F on $K \langle x_1, x_2 \rangle$ with respect to an operation is defined to be a linear functional on $K \langle x_1, x_2 \rangle$ such that $F(f \circ g) = F(f) F(g)$ for all $f, g \in K \langle x_1, x_2 \rangle$ where $\circ$ is a multiplication defined in $K \langle x_1, x_2 \rangle$ such that $f \circ g$ is again an entire function. Multiplication in $K \langle x_1, x_2 \rangle$ can be defined in two ways as follows:

(i) *Pointwise multiplication* : If $f, g \in K \langle x_1, x_2 \rangle$, then

$$(f \cdot g)(x_1, x_2) = f(x_1, x_2)\, g(x_1, x_2),\ x_1, x_2 \in K.$$

(ii) *Hadamard composition* : If $f, g \in K \langle x_1, x_2 \rangle$, then

$$(f \cdot g)(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} b_{mn} x_1^m x_2^n \quad \text{if} \quad x_1, x_2 \in K.$$

In relation to the above two operations, we have the following two theorems.

Theorem 1 — Let F be a function from $K \langle x_1, x_2 \rangle$ to K with $F \neq 0$. Then F is a continuous scalar homomorphism on $K \langle x_1, x_2 \rangle$ with respect to pointwise multiplication if and only if there exists a unique pair $(b, c) \in K \times K$ such that for all

$$f(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n \in K \langle x_1, x_2 \rangle$$

$$F(f) = f(b, c).$$

To prove the theorem, we need the following lemma :

Lemma — Given $\epsilon > 0$ and $(b, c) \in K \times K$, then there exists a $\delta > 0$ such that if $f, g \in K \langle x_1, x_2 \rangle$ and $d(f, g) < \delta$ implies $| f(b, c) - g(b, c) | < \epsilon$.

Proof of the Lemma : Given $\epsilon > 0$, let $R = \max [\, | b | , | c | \,]$. Choose $\delta > 0$ such that $\delta R < 1$. Let $f, g \in K \langle x_1, x_2 \rangle$ where

$$f(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n, \quad g(x_1, x_2) = \sum_{m, n=0}^{\infty} b_{mn} x_1^m x_2^n \,.$$

If $d(f, g) < \delta$ implies $| a_{mn} - b_{mn} | < \delta^{m+n}$.

Hence $\quad | f(b, c) - g(b, c) | \leqslant \sup | a_{mn} - b_{mn} | R^m R^n < (\delta R)^{m+n} \to 0$

as $m + n \to \infty$ since $\delta R < 1$.

Therefore $d(f, g) < \delta$ implies $| f(b, c) - g(b, c) | < \epsilon$

PROOF OF THEOREM 1 : Let F be a continuous scalar homomorphism on $K \langle x_1, x_2 \rangle$. By Theorem A, there exists a unique sequence $\{c_{mn}\}$ such that for all $f \in K \langle x_1, x_2 \rangle$, $F(f) = \sum_{m, n=0}^{\infty} a_{mn} c_{mn}$ where $\{ | c_{mn} |^{1/(m+n)}\}$ is bounded. For each m, n,

$$c_{mn} = F(x_1^m x_2^n) = [F(x_1)]^m [F(x_2)]^n.$$

Hence $c_{mn} = (C_{10})^m (C_{01})^n$. Using this we get

$$F(f) = \sum_{m, n=0}^{\infty} a_{mn} C_{10}^m C_{01}^n = f(C_{10}, C_{01})$$

where we can take $C_{10} = b$ and $C_{01} = c$.

Conversely let $(b, c) \in K \times K$. Define a function F from $K \langle x_1, x_2 \rangle$ to K as $F(f) = f(b, c)$. Then F is clearly a scalar homomorphism. Given $\epsilon > 0$, let $\delta > 0$ be such that if $f, g \in K \langle x_1, x_2 \rangle$, then $d(f, g) < \delta$. Then by Lemma 1,

$$| f(b, c) - g(b, c) | < \epsilon$$

so that F is continuous on $K \times K$.

Theorem 2 — A function $F : K \langle x_1, x_2 \rangle \to K$ is a continuous scalar homomorphism with respect to Hadamard composition if and only if

$$F(f) = \frac{1}{m!\, n!} \frac{\partial^{m+n} f(0)}{\partial x_1^m \partial x_2^n} \quad \text{for some } m \text{ and } n.$$

PROOF : Let F be a non-identically zero continuous linear functional on $K \langle x_1, x_2 \rangle$. Then there exists $\{C_{mn}\}$ such that for all $f(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n$ we have

$$F(f) = \sum_{m, n=0}^{\infty} a_{mn} C_{mn}.$$

Now consider $f(x_1, x_2) = x_1^m x_2^n$. Then we have $C_{mn}^2 = [F(f \cdot f)] = F(f) = C_{mn}$ for all m and n. Hence $C_{mn} = 0$ or $C_{mn} = 1$ for all m and n. If possible, let $C_{mn} = C_{pq} = 1$ for $p \neq m$ and $q \neq n$ for all m and n. Now let us consider $f(x_1, x_2) = x_1^m x_2^n + x_1^p x_2^q$. Then $4 = (C_{mn} + C_{pq})^2 = C_{mn} + C_{pq} = 2$ which is absurd. Thus atmost one C_{mn} is different from zero. Since $F \neq 0$, there is exactly one $C_{mn} \neq 0$ and $C_{pq} = 0$ for all $p \neq m$ and $q \neq n$. Hence we get

$$F(f) = a_{mn} = \frac{1}{m!\, n!} \frac{\partial^{m+n} f(0)}{\partial x_1^m \, \partial x_2^n}.$$

Conversely for fixed m and n, let us define

$$F(f) = \frac{1}{m!\, n!} \frac{\partial^{m+n} f(0)}{\partial x_1^m \, \partial x_2^n}.$$

Then F is obviously a continuous linear functional satisfying $F(f \,.\, g) = F(f) \,.\, F(g)$ for all $f, g \in K \langle x_1, x_2 \rangle$.

3. The Class of Bounded Linear Functionals on $K \langle x_1, x_2 \rangle$

Definition 2 — Let F be a linear functional on $K \langle x_1, x_2 \rangle$. Then F is said to be bounded if and only if there exists a $M > 0$ such that for all $f \in K \langle x_1, x_2 \rangle$, $F(f) \leqslant Md(f, 0)$ where 0 denotes the identically zero functional on $K \times K$.

It is easy to verify that a bounded linear functional on $K \langle x_1, x_2 \rangle$ is continuous. The following counter example shows that the converse may not be true.

Example — Let $C_{mn} = \pi^{m+n}$ where $|\pi| > 1$. Define a function from $K \langle x_1, x_2 \rangle$ to K by

$$F\left(\sum_{m,n=0}^{\infty} a_{mn} x_1^m x_2^n \right) = \sum_{m+n=1}^{\infty} \pi^{m+n} a_{mn}.$$

By Theorem A, F is a continuous linear functional. If F is bounded, there exists a constant M such that

$$\left| \sum_{m,n=0}^{\infty} \pi^{m+n} a_{mn} \right| \leqslant M \sup \left[|a_{00}|, |a_{mn}|^{1/(m+n)},\ m+n \geqslant 1 \right]$$

for all a_{mn} such that $|a_{mn}|^{1/(m+n)} \to 0$ as $m + n \to \infty$. Choose a $k > 1$ such that $|\pi|^k > \max[M, 2]$. Having chosen k as above, let $a_{0k} = \pi$ and $a_{mn} = 0$ if $(m, n) \neq (0, k)$. Then we have

$$\left| \sum_{m,n=0}^{\infty} \pi^{m+n} a_{mn} \right| = |\pi|^{k+1}.$$

But $M \sup [a_{00}, |a_{mn}|^{1/(m+n)}, m + n \geqslant 1] = M |\pi|^{1/k} < |\pi|^k |\pi|^{1/k} < |\pi|^{k+1}$ which is a contradiction. Therefore F defined above is not bounded.

However, the following theorem characterises bounded linear functionals on $K\langle x_1, x_2\rangle$.

Theorem 3 — Let B denote the class of bounded linear functionals on $K\langle x_1, x_2\rangle$. Let F be a function from $K\langle x_1, x_2\rangle$ to K. Then $F \in B$ if and only if there exists a unique (a, b, c) such that for all

$$f(x_1, x_2) = \sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n \in K\langle x_1, x_2\rangle,$$

$$F(f) = a_{00}a + a_{10}b + a_{01}c \quad \text{and} \quad \| F \| = \max\{ |a|, |b|, |c| \}$$

PROOF: Let $F \in B$. Then F is continuous so that there exists a unique sequence $\{b_{mn}\}$ such that

$$F\left(\sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n\right) = \sum_{m, n=0}^{\infty} a_{mn} b_{mn}$$

and

$$\left|\sum_{m, n=0}^{\infty} a_{mn} b_{mn}\right| \leqslant \| F \| \sup\{ |a_{00}|, |a_{mn}|^{1/(m+n)}, m+n \geqslant 1\}$$

for all sequences $\{a_{mn}\}$ for which $|a_{mn}|^{1/(m+n)} \to 0$ as $m + n \to \infty$. Suppose $b_{kj} \neq 0$ for some (k, j) with $k + j \geqslant 2$. Choose $a_{mn} = 0$ if $(m, n) \neq (k, j)$ and choose a_{kj} such that $|a_{kj}| < 1$ so that $|a_{kj}|^{(k+j-1)/(k+j)} < 1$. Since $|b_{kj}| \leqslant \| F \|$, we have $|a_{kj}|^{(k+j-1)/(k+j)} |b_{kj}| < \| F \|$. Then $|a_{mn}|^{1/(m+n)} \to 0$ as $m + n \to \infty$.

$$\left|\sum_{m, n=0}^{\infty} a_{mn} b_{mn}\right| = |a_{kj} b_{kj}| \leqslant \| F \| \, |a_{kj}|^{1/(k+j)}$$

so that $|a_{kj}|^{(k+j-1)/(k+j)} |b_{kj}| \leqslant \| F \|$ which is a contradiction. Hence $b_{kj} = 0$ if $k + j > 2$. So we have the following:

$$F\left(\sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n\right) = a_{00}b_{00} + a_{10}b_{10} + a_{01}b_{01}.$$

Therefore $$\left|F\left(\sum_{m, n=0}^{\infty} a_{mn} x_1^m x_2^n\right)\right| \leqslant \max(|b_{00}|, |b_{10}|, |b_{01}|)\, d(f, 0).$$

Hence $\| f \| \leqslant \max(|b_{00}|, |b_{10}|, |b_{01}|)$. To establish the equality, it is enough to establish the existence of a $g \in K\langle x_1, x_2\rangle$ such that

$$F(g_0) = \max(|b_{00}|, |b_{10}|, |b_{01}|).$$

Let $\max[|b_{00}|, |b_{01}|, |b_{10}|] = b_{ij} \neq 0, \ i, j = 0, 1.$

Let us define a_{pq} in the following manner

$$a_{pq} = \begin{cases} 1 \text{ if } p, q = i, j, \ i, j = 0, 1 \\ 0 \text{ if } p, q \neq i, j. \end{cases}$$

Let $g(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2$. Then $g(x_1, x_2) \in K \langle x_1, x_2 \rangle$ and

$$|f(g)| = |a_{00}b_{00} + a_{10}b_{01} + a_{01}b_{10}| = \max[|b_{00}|, |b_{01}|, |b_{10}|]$$

Conversely, given a, b, c, define a function from $K \langle x_1, x_2 \rangle$ to k by

$$F(f) = a_{00}a + a_{10}b + a_{01}c.$$

Then F is a bounded linear functional on $K \langle x_1, x_2 \rangle$.

Since $|F(f)| \leqslant \max\{|a|, |b|, |c|\} d(f, 0), \ F \in B.$

The following theorem gives the nature and the structure of the space B of bounded linear functionals on $K \langle x_1, x_2 \rangle$. The proof of the theorem is omitted, as it is obvious.

Theorem 4 — B is a non-archimedian Banach space with the non-archimedian norm of $F \in B$ as $\| F \| = \max\{|a|, |b|, |c|\}$. It is isometrically isomorphic to the non-archimedian Banach space $K \times K \times K$.

Acknowledgement

The author is thankful to the referee for many useful suggestions.

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