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A FAMILY OF BASIC HYPERGEOMETRIC AND COMBINATORIAL IDENTITIES AND CERTAIN SUMMATION FORMULAE

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In this paper certain illustrative new identities for basic hypergeometric series have been found by using an elementary method. These identities generalise certain known partition theoretic identities and indicate that these partition identities belong to a class of general basic hypergeometric identities.

1. INTRODUCTION

A number of recent advances in the theory of combinatorial identities and theory of partitions have made it desirable to have a look around for corresponding advances and obstacles in the theory of basic hypergeometric series, so that it has its impact on allied branches. Andrews (1974, 1975) in some recent communications has solved and posed certain problems in the theory of basic hypergeometric series.

A close look on some of the literature published in recent past shows that an inherent limitation in the development of basic hypergeometric investigations seems to be a lack of knowledge of summation formulae for such functions where the series have a general argument. The well-known summations

$$ {}_1\Phi_0(a; z) = \sum_{0}^{\infty} \frac{(a)_r}{(q)_r} z^r = \frac{(az)_\infty}{(z)_\infty} \qquad \ldots(1.1)$$

and

$$ {}_1\Psi_1(a; b; z) = \sum_{-\infty}^{\infty} \frac{(a)_r}{(b)_r} z^r = \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty} \qquad \ldots(1.2)$$

where $(a)_r = (1-a)(1-aq)\ \ldots\ (1-aq^{r-1}), (a)_0 = 1$ and

$(a)_{-r} = (-a)^{-r} q^{1/2r(r+1)}/(q/a)_r$ and $(a)_\infty = \lim_{r\to\infty} (a)_r,$

have been abundantly used and exploited by various authors e.g. Agarwal (1969), Andrews (1966, 1972, 1974, 1975, 1976a) and Varma (1977), besides others. Their

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main objective, however, in using the above two summations has been to replace certain products occurring in a particular problem by their equivalent ${}_1\Phi_0$ or a ${}_1\Psi_1$ and then obtain the desired result by a change in the order of summation or by some suitable manipulation. Nevertheless, their systematic use in finding classes of more general summation formulae where the series may have a general argument seems to have received little attention (although they have been used to obtain a number of summations where the series have particular parameters and/or particular arguments). Obviously, any such attempt to find summation formulae for basic hypergeometric series with general arguments must heavily weigh upon the generality of the parameters in some fashion or the other, as for instance, Andrews (1976b) (see also Andrews *et al.* 1972) has recently given two general summation theorems involving reciprocal polynomials but both of them are terminating and with special type of parameters.

The theme of this paper is based on a simple observation which has lead to using (1.1) and (1.2) from the standpoint of finding classes of new identities and summation formulae, where either the argument is a general one or the method indicates the extent to which some of the existing summation formulae could be extended to similar ones for higher series or for more general arguments.

One finds that in many of the important known basic hypergeometric summation formulae two of the numerator parameters are of the type $q\sqrt{a}$, $-q\sqrt{a}$ and the corresponding denominator parameters are $\sqrt{a}$, $-\sqrt{a}$ (the "very well-poised" terms of special kind). Thus, the numerator parameters are q times the denominator parameters. This appears to be a very crucial selection of parameters which can be made to give (1.1) and (1.2) classes of more general summation formulae for series of higher orders, with general arguments.

This basic idea has been developed in section 2 and does not appear to have been noted explicitly in the form it has been used, herein, although used abundantly in an implicit manner.

In section 3 we deduce families of basic hypergeometric identities each depending upon the use of (2.1) or (2.2). In section 4 some results of section 3 are shown to give more general known combinatorial identities and also their further generalisations which follow in a very natural manner from the method used herein. Finally, in section 5 the idea of section 2 has been used to sum certain bilateral basic hypergeometric series ${}_4\Psi_4$ and ${}_5\Psi_5$ and it has been shown why further extensions of this type are not possible to similar series of higher orders.

The central idea behind this communication is not to give an exhaustive list of all families of identities that could possibly be found by this method but only to indicate the possible variety of obtainable classes.

In what follows we use the notations

$${}_r\Phi_s[a_1, a_2 \ldots, a_r; b_1, b_2, \ldots, b_s; t] \doteq \sum_{0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_r)_n t^n}{(q)_n(b_1)_n \ldots (b_s)_n}$$

and

$${}_s\Psi_s[a_1, a_2, \ldots, a_s; b_1, b_2, \ldots, b_s; t] = \sum_{-\infty}^{\infty} \frac{(a_1)_n \ldots (a_s)_n}{(b_1)_n \ldots (b_s)_n} t^n.$$

2. The Main Theorems

Theorem I — Subject to appropriate conditions of convergence,

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(q)_n} \prod_{s=1}^{l} \frac{(a_s q)_n}{(a_s)_n} t^n = \frac{(\alpha t)_\infty}{(t)_\infty} \prod_{s=1}^{l} (1 - a_s)^{-1} \sum_{r=0}^{l} B_r \frac{(t)_r}{(\alpha t)_r} \quad \ldots(2.1)$$

where

$$\prod_{s=1}^{l} (1 - a_s q^n) \equiv \sum_{r=0}^{l} B_r q^{rn}.$$

Theorem II — Subject to appropriate conditions of convergence,

$$\sum_{n=-\infty}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \prod_{s=1}^{l} \frac{(a_s q)_n}{(a_s)_n} t^n = \frac{(\beta/\alpha)_\infty (q)_\infty}{(q/\alpha)_\infty (\beta)_\infty} \prod_{s=1}^{l} (1 - a_s)^{-1}$$

$$\times \frac{(\alpha t)_\infty (q/\alpha t)_\infty}{(t)_\infty (\beta/\alpha t)_\infty} \sum_{r=0}^{l} B_r \frac{(t)_r}{(q\alpha t/\beta)_r} (q/\beta)^r \quad \ldots(2.2)$$

where $\prod_{s=1}^{l} (1 - a_s q^n) = \sum_{r=0}^{l} B_r q^{rn}$.

Proof of Theorem I : We have

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(q)_n} \prod_{s=1}^{l} \frac{(a_s q)_n}{(a_s)_n} t^n = \prod_{s-1}^{l} (1 - a_s)^{-1} \sum_{0}^{\infty} \frac{(\alpha)_n}{(q)_n} t^n \sum_{r=0}^{l} B_r q^{rn}$$

$$= \prod_{s=1}^{l} (1 - a_s)^{-1} \sum_{r=0}^{l} B_r \frac{(\alpha t q^r)_\infty}{(t q^r)_\infty}, \text{ by (1.1).}$$

Simplication gives (2.1).

Similarly, one can prove (2.2), where again (1.1) and (1.2) have been used.

3. Applications of (2.1) and (2.1)

(a) As a simple illustration consider the sum of the series

$$ {}_3\Phi_2\left[\begin{matrix} a, bq, cq; t \\ b, c \end{matrix}\right]. $$

Theorem I gives, on simplification, that

$$ {}_3\Phi_2\left[\begin{matrix} a, bq, cq; t \\ b, c \end{matrix}\right] = \frac{(at)_\infty}{(t)_\infty (1-b)(1-c)} \left[1 - (b+c)\frac{1-t}{1-at} + bc\frac{(1-t)(1-tq)}{(1-at)(1-atq)}\right]. \qquad \ldots(3.1) $$

It is interesting to compare the summation (3.1) with a similar result of Andrews (1976a), namely, the sum of ${}_3\Phi_2\,[a, b, cq; d, c; d/abq]$. This for $d = b/q$ can be summed by (3.1) but not with a general argument. In the result (3.1) in order to maintain the generality of the argument we have sacrificed the generality of only the denominator parameters, whereas, in the result of Andrews (1976a) to maintain the generality of the denominator parameters the argument has been made to depend upon the parameters.

Obviously, one could go on augmenting any number of numerator and denominator parameters of the type αq and α respectively, to obtain the sum of higher series.

(b) As in (a) above, from Theorem I, we get the sum of a

$$ {}_4\Phi_3\left[\begin{matrix} a, dq, eq, fq; t \\ d, e, f \end{matrix}\right] \text{ equal to} $$

$$ \frac{1}{(1-d)(1-e)(1-f)}\frac{(at)_\infty}{(t)_\infty}\left[1-(d+e+f)\frac{1-t}{1-at} + (de+ef+df)\frac{(1-t)(1-tq)}{(1-at)(1-atq)} - def\frac{(1-t)(1-tq)(1-tq^2)}{(1-at)(1-atq)(1-atq^2)}\right]. \qquad \ldots(3.2) $$

For $f = \sqrt{a}$, $e = -\sqrt{a}$, we get the sum of the following non-terminating, nearly-poised, ${}_4\Phi_3$ with special types of 'well-poised' parameters and with a general argument not usually mentioned in literature, namely,

$$ {}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, bq; t \\ \sqrt{a}, -\sqrt{a}, b \end{matrix}\right] = \frac{(at)_\infty}{(t)_\infty}\frac{1}{(1-a)(1-b)}\left[\sum_{r=0}^{3} B_r \frac{(t)_r}{(\alpha t)_r}\right] \qquad \ldots(3.3) $$

with $B_0 = 1$, $B_1 = -b$, $B_2 = -a$ and $B_3 = ab$.

(c) As an application of (2.2), we have

$$
{}_3\Psi_3\left[\begin{matrix}\alpha, & aq, & bq; & t\\ \beta, & a, & b & \end{matrix}\right]
$$

$$
= \frac{1}{(1-a)(1-b)} \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(q/\alpha)_\infty (\beta)_\infty (t)_\infty (\beta/\alpha t)_\infty} \sum_{r=0}^{2} B_r \frac{(t)_r}{(q\alpha t/\beta)_r} (q/\beta)^r \qquad \ldots(3.4)
$$

with $B_0 = 1$, $B_1 = -(a+b)$, $B_2 = ab$.

(d) As still another application consider the sum of the family of series ${}_2\Psi_2\,[\alpha, aq^m; \beta, a; t]$, where m is a positive integer.
(2.2) gives

$$
{}_2\Psi_2\,[\alpha, aq^m; \beta, a; t] = \frac{1}{(a)_m} \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(\beta)_\infty (q/\alpha)_\infty (t)_\infty (\beta/\alpha t)_\infty}
$$

$$
\times \sum_{r=0}^{m} B_r(a) \frac{(t)_r}{(q\alpha t/\beta)_r} (q/\beta)^r \qquad \ldots(3.5)
$$

where $B_r(a) = \dfrac{(q^{-m})_r}{(q)_r} a^r q^{mr}$, a constant times the Gaussian polynomial.

In the next section we shall give a discussion of some of the very interesting limiting cases of (3.5) which yield a family of known combinatorial identities and their generalizations.

(e) As still another illustration, we could introduce terms with different bases both in (2.1) and (2.2). For instance, for m, a nonnegative integer, from Theorem I, we have

$$
\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(q)_n} \frac{(aq^3)_{n;\,q^3}}{(a)_{n;\,q^3}} t^n = \frac{1}{1-a}\left[\frac{(\alpha t)_\infty}{(t)_\infty} - \frac{a(\alpha t q^3)_\infty}{(tq^3)_\infty}\right].
$$

Extending the argument one could find a transformation for

$$
\sum_{n=0}^{\infty} \frac{(\alpha)_n\,(aq^{3m})_{n;\,q^3}}{(q)_n\,(a)_{n;\,q^3}} t^n.
$$

4. Certain Limiting Cases of (3.5) and Families of Combinatory and Hypergeometric Identities

The summation formula (3.5) gives

$$
{}_2\Psi_2\,[\alpha, aq^m; \beta, a; t] = \frac{1}{(a)_m} \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(\beta)_\infty (q/\alpha)_\infty (t)_\infty (\beta/\alpha t)_\infty} \times
$$

(*equation continued on p. 733*)

$$\times \sum_{r=0}^{m} \frac{(q^{-m})_r}{(q)_r}, \; a^r q^{mr} \frac{(t)_r}{(q\alpha t/\beta)_r} (q/\beta)^r. \quad \ldots(4.1)$$

If let $a \to q$ in (4.1), we get the family of identities,

$$ {}_2\Phi_1(\alpha, q^{m+1}; \beta; t) = \frac{1}{(q)_m} \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(\beta)_\infty (q/\alpha)_\infty (t)_\infty (\beta/\alpha t)_\infty}$$

$$\times {}_2\Phi_1(q^{-m}, t; tq\alpha/\beta; q^{2+m}/\beta)$$

$$+ (-)^{m+1} \frac{(q/\beta)_{m+1}}{(q/\alpha)_{m+1}} q^{-1/2m(m+1)}(\beta/\alpha t)^{m+1}$$

$$\times {}_2\Phi_1 \left[\begin{matrix} q^{2+m}/\beta, q^{1+m}; \beta q^{-m}/\alpha t \\ q^{2+m}/\alpha \end{matrix} \right]. \quad \ldots(4.2)$$

Reversing the first ${}_2\Phi_1$ on the right and rewriting the second ${}_2\Phi_1$ on the right of (4.2), by means of Heine's fundamental transformation, namely,

$$ {}_2\Phi_1 \left(\begin{matrix} \alpha, \beta; t \\ \gamma \end{matrix} \right) = \frac{(\beta t)_\infty (\gamma/\beta)_\infty}{(t)_\infty (\gamma)_\infty} {}_2\Phi_1 \left(\begin{matrix} \beta, \alpha\beta t/\gamma; \gamma/\beta \\ \beta t \end{matrix} \right),$$

we get the transformation

$$ {}_2\Phi_1 [\alpha, q^{m+1}; \beta; t] = \frac{(-)^m q^{1/2m(m+1)}}{(q)_m} (q/\beta)^m \frac{(t)_m}{(tq\alpha/\beta)_m}$$

$$\times \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(q/\alpha)_\infty (\beta)_\infty (t)_\infty (\beta/t\alpha)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-m}, \beta q^{-m}/t\alpha; \alpha \\ q^{1-m}/t \end{matrix} \right]$$

$$+ \frac{(q/\beta)_{m+1}}{(\alpha t/\beta)_{m+1}} {}_2\Phi_1 \left[\begin{matrix} q^{1+m}, q/t; q/\alpha \\ q\beta/\alpha t \end{matrix} \right]. \quad \ldots(4.3)$$

(4.3) is the desired transformation which gives many known and new families of combinatory identities and summation formulae which were not apparent to the initial authors because of their specific mode of deduction of their identities. It extends a result of Varma (1977) which is obtained by taking $t = q$, namely

$$ {}_2\Phi_1 \left[\begin{matrix} \alpha, q^{m+1}; q \\ \beta \end{matrix} \right] = \frac{(\alpha)_\infty}{(\beta)_\infty} \frac{\alpha^{-m-1}}{(\beta q^{-m-1}/\alpha)_{m+1}}$$

$$\times \left\{ \frac{(\beta q^{-m-1})_\infty}{(\alpha)_\infty} - \sum_{r=0}^{m} \frac{(\beta q^{-m-1}/\alpha)}{(q)_r} \alpha^r \right\}. \quad \ldots(4.4)$$

Special Cases

(i) For $m = 0$, (4.3) gives

$$\sum_{0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} t^n = \frac{(\beta/\alpha)_\infty (q)_\infty (\alpha t)_\infty (q/\alpha t)_\infty}{(q/\alpha)_\infty (\beta)_\infty (t)_\infty (\beta/\alpha t)_\infty} + \frac{(1-(q/\beta))}{(1-(\alpha t/\beta))}$$

$$\times\ {}_2\Phi_1(q, q/t; q\beta/\alpha t; q/\alpha) \qquad \ldots(4.5)$$

which is an extension of a result of Andrews *et al.* [1972, (4.1)], obtained by taking $t = q$.

As already remarked by Andrews *et al.* (1972) (4.5) extends the family of combinatorial identities (1.1) and (1.5) of Murray Eden (1968).

(ii) For $t = q^h$ (h a nonnegative integer), $\beta = 0$, we get from (4.5) the family of combinatorial identities of Murray Eden [1968, (1)]. The generalization of this has not been given by Andrews *et al.* (1972).

(iii) For $t = q^h$ (h, a nonnegative integer), $\alpha = 0$, we get from (4.5) the family of combinatorial identities (2.2) proved by Andrews *et al.* (1972).

(iv) In (4.3) and (4.5), if we take $t = q^k$ (k, a nonnegative integer), we get an infinite sequence of transformations for a particular ${}_2\Phi_1$ of the type ${}_2\Phi_1\,[\alpha, q^{m+1}; \beta; q^k]$, a fact not obvious from the result of Varma (1977) or from the method of deduction of Andrews *et al.* (1972). In particular, if we take $t = q^2$, we get a particularly simple summation, namely

$${}_2\Phi_1(\alpha, q^{m+1}; \beta; q^2) = \frac{(-)^m q^{1/2m(m+1)}}{(q\alpha/\beta)_{m+2}} (q/\beta)^{m+2} \frac{(\alpha)_\infty}{(\beta)_\infty}$$

$$\times \left\{ \sum_{r=0}^{m} (\beta q^{-m-2}/\alpha)_r (\alpha q)^r - q^{m+1} \sum_{r=0}^{m} (\beta q^{-m-2}/\alpha)_r \alpha^r \right\}$$

$$+ \left\{ (q/\beta)_{m+1}/(\alpha q^2/\beta)_{m+1} \right\} \left(1 + \frac{q(1-q^{m+1})}{(\beta - \alpha q)} \right). \qquad \ldots(4.6)$$

For $m = 0$, we get

$$\sum_{0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} q^{2n} = \frac{q^2/\beta^2}{(1-(q\alpha/\beta))(1-(q^2\alpha/\beta))} \frac{(\alpha)_\infty}{(\beta)_\infty} \left\{1 - q\right\}$$

$$+ \left(\frac{\beta - q}{\beta - \alpha q^2} \right) \left(1 + \frac{q(1-q)}{(\beta - \alpha q)} \right). \qquad \ldots(4.7)$$

(v) From (4.4) and (4.6), following the technique of Theorem I, it is easy to sum the series

$${}_3\Phi_2\,[\alpha, q^{m+1}, \gamma q; \beta, \gamma; q].$$

Since

$$ {}_3\Phi_2\,[\alpha,\, q^{m+1},\, \gamma q;\, \beta,\, \gamma;\, q] = \frac{1}{1-\gamma}\,\{{}_2\Phi_1(\alpha,\, q^{m+1};\, \beta;\, q\} $$

$$ - \gamma\,{}_2\Phi_1(\alpha,\, q^{m+1};\, \beta;\, q^2)\} $$

summing the ${}_2\Phi_1$'s by (4.4) and (4.6), we get the required sum. It is easy to see that one could easily continue this method to obtain the sum of any series

$$ {}_{s+2}\Phi_{s+1}\,[\alpha,\, q^{m+1},\, (\gamma_s)q;\, \beta,\, (\gamma_s);\, q^k], $$

k a nonnegative integer.

$m = 0$ in these sums will give an entire family of identities of the type given by Andrews *et al.* (1972).

The results of this section are particularly important and interesting in view of a recent comment of Andrews, made in connection with Varma's result in (4.4). "It is interesting that this identity has assumed considerable importance in the study of Ramanujan's 'lost' note book especially with respect to partial theta functions" [*Mathematical Reviews* 57, (1979), # 12932, p. 1680].

5. Certain Summation Formulae for Basic Bilateral Series

In this section we use the method of the previous sections to obtain certain sums of basic bilateral ${}_4\Psi_4$ and ${}_5\Psi_5$ series, which we believe are new.

(i) *The sum of a* ${}_4\Psi_4$ — We know that (Shukla 1959)

$$ \Psi\left(\frac{1}{a}\right) \equiv {}_3\Psi_3\begin{bmatrix} a,\, b,\, cq;\, 1/a \\ d,\, bq,\, c \end{bmatrix} = \frac{(d/b)_\infty\,(q)_\infty\,(q)_\infty\,(bq/a)_\infty}{(q/b)_\infty\,(d)_\infty\,(q/a)_\infty\,(bq)_\infty}\left(\frac{c-b}{c-1}\right) \qquad \ldots(5.1) $$

and

$$ \Psi\left(\frac{q}{a}\right) \equiv {}_3\Psi_3\begin{bmatrix} a,\, b,\, cq;\, q/a \\ d,\, bq,\, c \end{bmatrix} = \frac{(d/b)_\infty\,(q)_\infty\,(q)_\infty\,(bq/a)_\infty}{(q/b)_\infty\,(d)_\infty\,(q/a)_\infty\,(bq)_\infty}\,\frac{1}{b}\,\frac{(c-b)}{1-c}. \qquad \ldots(5.2) $$

It is easy to see that

$$ {}_4\Psi_4\begin{bmatrix} a,\, b,\, cq,\, eq;\, 1/a \\ d,\, bq,\, c,\, e \end{bmatrix} $$

$$ = \frac{1}{1-e}\,\Psi\left(\frac{1}{a}\right) - \frac{e}{1-e}\,\Psi(q/a), $$

which gives that

$$ {}_4\Psi_4\begin{bmatrix} a,\, b,\, cq,\, eq;\, 1/a \\ d,\, bq,\, c,\, e \end{bmatrix} = \frac{(c-b)\,(1-e/b)}{(c-1)\,(1-e)}\,\frac{(d/b)_\infty\,(q)_\infty\,(q)_\infty\,(bq/a)_\infty}{(q/b)_\infty\,(d)_\infty\,(q/a)_\infty\,(bq)_\infty}. \qquad \ldots(5.3) $$

A little calculation will further show that it is not possible to obtain the sum of a similar ${}_5\Psi_5$ or a series of higher order, from (5.3).

(ii) *The sum of a* ${}_5\Psi_5$ — We know from a result of Bailey (1950) that

$$ {}_3\Psi_3(e) \equiv {}_3\Psi_3\left[\begin{matrix} e, f, g; q/efg \\ q/e, q/f, q/g \end{matrix}\right] = \frac{(q)_\infty (q/ef)_\infty (q/eg)_\infty (q/fg)_\infty}{(q/e)_\infty (q/f)_\infty (q/g)_\infty (q/efg)_\infty} \qquad \ldots(5.4) $$

and

$$ {}_3\Psi_3\left[\begin{matrix} e, f, g; q^2/efg \\ q^2/e, q^2/f, q^2/g \end{matrix}\right] = \frac{(q)_\infty (q^2/ef)_\infty (q^2/eg)_\infty (q^2/fg)_\infty}{(q^2/e)_\infty (q^2/f)_\infty (q^2/g)_\infty (q^2/efg)_\infty}. \qquad \ldots(5.5) $$

We can use (5.4) and (5.5) to obtain the sums of two ${}_5\Psi_5$ series where the product of the corresponding numerator and parameters are respectively q and q^2. In particular,

$$ {}_5\Psi_5\left[\begin{matrix} e, f, g, c, q^2/c; 1/efg \\ q/e, q/f, q/g, q/c, c/q \end{matrix}\right] $$

$$ = \sum_{-\infty}^{\infty} \frac{(e)_n (f)_n (g)_n}{(q/e)_n (q/f)_n (q/g)_n} \left(\frac{1}{efg}\right)^n \frac{1 - cq^{n-1}}{1 - (c/q)} \cdot \frac{1 - q^{n+1}/c}{1 - (q/c)}. $$

$$ = \frac{1}{(1 - (c/q))(1 - (q/c))} \left[\sum_{-\infty}^{\infty} \frac{(e)_n (f)_n (g)_n}{(q/e)_n (q/f)_n (q/g)_n} \left(\frac{1}{efg}\right)^n \right. $$

$$ \left. \times \left\{ (1 - (eq^n))(1 - (q^n/e)) + q^n \left(e + \frac{1}{e} - \frac{q}{c} - \frac{c}{q} \right) \right\} \right] $$

$$ = \frac{1}{(1 - (c/q))(1 - (q/c))} \left[\left(e + \frac{1}{e} - \frac{q}{c} - \frac{c}{q} \right) {}_3\Psi_3(e) \right. $$

$$ \left. + (1 - e)\left(1 - \frac{1}{e}\right) {}_3\Psi_3(eq) \right]. $$

Simplification gives that

$$ {}_5\Psi_5\left[\begin{matrix} e, f, g, c, q^2/c; 1/efg \\ q/e, q/f, q/g, q/c, c/q \end{matrix}\right] $$

$$ = \left\{ 1 - \frac{(1 - e)(1 - f)(1 - g)}{(1 - (q/c))(1 - (c/q))(1 - (efg))} \right\} \frac{(q)_\infty (q/ef)_\infty (q/eg)_\infty (q/fg)_\infty}{(q/e)_\infty (q/f)_\infty (q/g)_\infty (q/efg)_\infty}. \qquad \ldots(5.6) $$

Proceeding exactly as above and now using (5.5), we get the following sum of a ${}_5\Psi_5$, where the product of the corresponding numerator and denominator parameters is q^2, namely,

$$
{}_5\Psi_5\left[\begin{matrix} e, f, g, c, q^3/c; q/efg \\ q^2/e, q^2/f, q^2/g, q^2/c, c/q \end{matrix}\right]
$$

$$
= \left\{1 - \frac{(1-e)(1-f)(1-g)}{(1-(q^2/c))(1-(c/q))(1-(efg/q))}\right\}
$$

$$
\times \frac{(q)_\infty (q^2/ef)_\infty (q^2/eg)_\infty (q^2/fg)_\infty}{(q^2/e)_\infty (q^2/f)_\infty (q^2/g)_\infty (q^2/efg)_\infty}. \quad \ldots(5.7)
$$

A natural question arises that whether there do exist summation formulae, for other ${}_5\Psi_5$ series where the product of the corresponding numerator and parameter terms is q^3, q^4 etc., and if there are similar series of higher order, with particular forms of parameters that can be summed by the help of (5.3), (5.6) and (5.7). Unfortunately, the methods of this paper do not give any further extensions and nor do I believe that they exist except for the trivial ones obtained by taking out a few terms common.

It may finally be remarked that (2.1) and (2.2) can be made to sum up terminating series also, if we choose say $\alpha = q^{-N}$, N, a nonnegative integer. In fact, all the results of Varma [1977; (i), (ii), (iii), p. 373] for finite summation formula of basic hypergeometric series with one or two bases can be obtained from our theorems. I hope to use some of the results of this paper in a subsequent communication to obtain new transformations and summations.

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