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DECOMPOSABLE AND SPECTRAL OPERATORS

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In the present paper the author finds a necessary and sufficient condition for a decomposable operator on a Banach space to be a spectral operator. Also conditions are obtained under which a decomposable operator on a Hilbert Space is of the form (i) Self adjoint plus a commuting quasi-nilpotent operator. (ii) Unitary plus a commuting quasi-nilpotent operator.

1. INTRODUCTION

The notion of a decomposable operator on a Banach space introduced by Foias (1968) can be viewed as a nontrivial generalization of the notion of spectral operators which was introduced and studied by Dunford (1954). Hence one would like to find conditions under which a decomposable operator becomes spectral. Wadhwa (1973) has found a necessary and sufficient condition for a decomposable operator on a Hilbert space to be equal to the sum of a normal plus a commuting quasi-nilpotent operator (hence spectral). We obtain here conditions for a decomposable Banach space operator to be spectral.

We begin with introducing some notations and definitions which are needed in the sequel. X will be used to denote a complex Banach space and H will denote a complex Hilbert space. X^* will be the dual of X. By $B(X)$ ($B(H)$) we denote the Banach algebra of operators on X (on H). For $T \in B(X)$, $N(T)$ and $R(T)$ will denote its null space and range respectively and T' will be its conjugate. If $T \in B(H)$ then T^* is its adjoint.

Complex plane will be denoted by C and for any $\delta \subseteq C$, $\bar{\delta}$ is its closure and δ' is its complement. For $M \subseteq X$, M° is the annihilator of M in X^* and for $N \subseteq X^*$, ${}^\circ N$ is the annihilator of N in X.

Let $T \in B(X)$. A subspace M of X is called a spectral maximal subspace for T if (i) M is invariant under T. (ii) if N is any subspace of X invariant under T and $\sigma(T/N) \subseteq \sigma(T/M)$ then $N \subseteq M$, where T/N denotes the restriction of T to the subspace N and $\sigma(T)$ denotes the spectrum of T.

Any $T \in B(X)$ is called a decomposable operator if for every finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there exists a family $\{M_i\}_{i=1}^n$ of spectral maximal subspaces of T such that

(i) $X = \sum_{i=1}^{n} M_i$ and (ii) $\sigma(T/M_i) \subseteq G_i$ for $i = 1, 2, 3, \dots n.$

For results on spectral operators we refer to Dunford and Schwartz (1971).

An operator $T \in B(X)$ is said to have the single valued extension property if for any analytic function $f : D(f) \to X$ where $D(f)$ is an open subset of complex plane with $(T - zI) f(z) \equiv 0$, we have $f(z) \equiv 0$. This definition is due to Colojoara and Foias (1968) and is equivalent to one given by Dunford and Schwartz (1971).

For an operator $T \in B(X)$ with single valued extension property, if we consider the set $\rho(T, x)$ of all $z \in C$ such that there exists a neighbourhood of z and a vector valued analytic function $z_1 \to x(z_1)$ defined on this neighbourhood, which satisfies $(T - z_1 I)\, x(z_1) = x$, then $\rho(T, x)$ is called resolvent set of T at x and the complement of $\rho(T, x)$ is called spectrum of T at x and is denoted by $\sigma(T, x)$. Since T has single valued extension property the function $x(z_1)$ is unique. For any $T \in B(X)$ with single valued extension property and δ in the complex plane define

$$\mathrm{x}_T(\delta) = \{x \in X \mid \sigma(T, x) \subseteq \delta\}$$

(Note that the definition of $\rho(T, x)$ makes sense even for operators without single valued extension property. (But in this case the above function $x(z_1)$ may not be unique.) Hence $\mathrm{x}_T(\delta)$ can be defined for any $T \in B(X)$.) For these concepts see Colojoara and Foias (1968).

If $T_1, T_2 \in B(X)$ then T_1 is said to be quasi-nilpotent equivalent to T_2 if

$$\lim_{n\to\infty} \| (T_1 - T_2)^{[n]} \|^{1/n} = 0$$

and

$$\lim_{n\to\infty} \| (T_2 - T_1)^{[n]} \|^{1/n} = 0$$

where

$$(T_1 - T_2)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_1^k\, T_2^{n-k}.$$

[For this concept refer to Colojoara and Foias (1968)].

2. Main Results

Theorem 2.1 — Suppose X is a Banach space and $T \in B(X)$ is a decomposable operator. Then T is spectral if and only if there exists a bounded family $\{E(\delta) \mid \delta$ closed in $C\}$ of projections with $R(E(\delta)) = \mathrm{x}_T(\delta)$, $\sigma(T, E(\delta)\, x) \subseteq \sigma(T, x)$ for each

$x \in X$ and δ closed in C and

$$R(E'(\delta)) = \cap \{R(E'(\bar{G})) \mid G \supseteq \delta \text{ and } G \text{ is open}\}.$$

PROOF : If T is spectral, then the family $\{E(\delta)\}$ exists, where E is the resolution of identity for T, $R(E(\delta)) = \mathbf{x}_T(\delta)$ (Dunford and Schwartz 1971, Theorem 4, p. 1934) the family $\{E(\delta)\}$ is bounded (Dunford and Schwartz 1971, Corollary 4, p. 1931) and $\sigma(T, E(\delta)\, x)) \subseteq \sigma(T, x)$ as $E(\delta)$ commutes with T by definition.

Now T' is also decomposable and thus for any closed set δ

$$\mathbf{x}_{T'}(\delta) = \cap \{\mathbf{x}_{T'}(\bar{G}) \mid G \text{ is open and } G \supset \delta\}$$

(see Foias 1968). But T' is spectral also and E' is its resolution of identity (Dunford and Schwartz 1971, Lemma 6, p. 2148). Hence $\mathbf{x}_{T'}(\delta) = R(E'(\delta))$ for any closed set δ. This completes the necessity.

For the sufficiency part we shall use the terminology and notations given by Dunford and Schwartz (1971). It follows from Corollary 1.4 and Theorem 1.4 of Colojoara and Foias (1968, p. 31) that T satisfies Dunford's condition (A) and (C). To show that T is spectral we have only to show that T satisfies conditions (B) and (D). First we show that T sastisfies condition (B).

If P is a projection and $x \in R(P)$ and $y \in N(P)$, then

$$\| x \| = \| P(x + y) \| \leqslant \| P \| \| x + y \|$$

Now let $\sigma(T, x) = \delta_1$ and $\sigma(T, y) = \delta_2$ with $\delta_1 \cap \delta_2 = \phi$ then by using the property $\sigma(T, E(\delta_1)\, y) \subseteq \sigma(T, y)$, it can be shown that $y \in N(E(\delta_1))$. Thus

$$\| x \| \leqslant \| E(\delta_1) \| \| x + y \| \leqslant K \| x + y \|$$

where K is the bound of the family $\{E(\delta) \mid \delta \text{ closed in } C\}$. Thus T has the property (B).

To show the T has the property (D), we first show that

$$X = \mathbf{x}_T(\delta) \oplus \overline{\mathbf{x}_T(\delta')} \quad \text{for each closed set } \delta.$$

As $X = R(E(\delta)) \oplus N(E(\delta))$, it is enough to show that

$$N(E(\delta)) = \overline{\mathbf{x}_T(\delta')}.$$

It is easily seen that $N(E(\delta)) \supseteq \overline{\mathbf{x}_T(\delta')}$.

To show the reverse inclusion, we first show that

$$N(E(\bar{G})) \subseteq \mathbf{x}_T(\delta') \quad \text{for any open set } G \text{ containing } \delta.$$

The proof of this is essentially same as given by Wadhwa (1973). We reproduce it here for the sake of completeness. Since G and δ' form an open cover of $\sigma(T)$, the decomposibility of T gives that $X = Z_1 + Z_2$ where Z_1 and Z_2 are spectral maximal subspaces of T such that $\sigma(T/Z_1) \subseteq G \cap \sigma(T)$ and $\sigma(T/Z_2) \subseteq \delta' \cap \sigma(T)$. If $x \in N(E(\bar{G}))$, then $E(\bar{G})\,x = 0$. But $x = z_1 + z_2$, $z_i \in Z_i$ $(i = 1, 2)$. Hence $z_1 + E(\bar{G})\,z_2 = 0$. Thus $\sigma(T, z_1) \subseteq \sigma(T, z_2) \subseteq \delta'$ and $\sigma(T, x) \subseteq \sigma(T, z_1) \cup \sigma(T, z_2) \subseteq \delta'$. So $N(E(\bar{G}) \subseteq \mathbf{x}_T(\delta')$. Taking annihilators,

$$(\mathbf{x}_T(\delta'))^\circ \subseteq (N(E(\bar{G}))^\circ$$

$$= R(E'(\bar{G}))$$

[By Ex. 22, p. 514 of Dunford and Schwartz (1957)].

Thus $(\mathbf{x}_T(\delta'))^\circ \subseteq \cap \{R(E'(\bar{G})) \mid G \supseteq \delta \text{ and } G \text{ is open}\}$. Our hypothesis now yields

$$(\mathbf{x}_T(\delta'))^\circ \subseteq R(E'(\delta)) = \Big(N(E(\delta))\Big)^\circ.$$

Now again taking annihilators we get

$$N(E(\delta)) \subseteq \overline{\mathbf{x}_T(\delta')}.$$

Thus $\quad X = \mathbf{x}_T(\delta) \oplus \overline{\mathbf{x}_T(\delta')}$ for each closed set δ.

This puts each closed set δ in $S_1(T)$. So there exists a projection $E_1(\delta)$ for each closed set which will coincide with $E(\delta)$ because of uniqueness property of E_1 (see Dunford and Schwartz 1971, p. 2138). Also $E_1(\delta')$ is defined for each closed set δ as $S_1(T)$ is closed under complementation. If δ is closed, then $\delta' = \cup\, \delta_n$ where $\{\delta_n\}$ is an increasing sequence of closed sets. If $x \in X$, then $x = x_1 + x_2$ where $x_1 \in \mathbf{x}_T(\delta)$ and $x_2 = \lim_{m\to\infty} y_m$; $y_m \in \mathbf{x}_T(\delta')$ for all m. If we show that

$$x = \lim_{n\to\infty} (E(\delta)\,x + E(\delta_n)\,x)$$

then each closed set δ is in $S(T)$ and condition (D) will be satisfied. Since $E(\delta)\,x = x_1$, it is enough to show that

$$x_2 = \lim_{n\to\infty} E(\delta_n)\,x$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} E(\delta_n)\,y_m.$$

Now $\quad \lim_{m\to\infty} \lim_{n\to\infty} E(\delta_n)\,y_m = \lim_{m\to\infty} y_m$

as there exists a positive integer n_0 such that

$$\sigma(T, y_m) \subseteq \delta_n \quad \text{for} \quad n \geqslant n_0$$

Thus $$\lim_{m\to\infty} \lim_{n\to\infty} E(\delta_n)\, y_m = x_2.$$

Hence the proof is over if we are allowed to change the order in taking limits. For this we use Theorem 5.4 of Munroe (1959, p. 42). As given there, we metrize the set N of positive integers together with the point ∞ by defining

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right| \quad \text{and} \quad d(\infty, n) = \frac{1}{n}\cdot$$

Also we define a function f from $(N \cup \{\infty\}) \times (N \cup \{\infty\})$ to X by

$$\begin{aligned} f(n, m) &= E(\delta_n)\, y_m \quad &&\text{if} \quad n \neq \infty,\ m \neq \infty \\ &= E(\delta_n)\, x_2 &&\text{if} \quad n \neq \infty,\ m = \infty \\ &= y_m &&\text{if} \quad n = \infty,\ m \neq \infty \\ &= E_1(\delta')\, x &&\text{if} \quad n = \infty,\ m = \infty. \end{aligned}$$

It can be verified that this function f satisfies all the required conditions of Theorem 5.4 of Munroe (1959). Hence both the iterated limits $\lim_{n\to\infty} \lim_{m\to\infty} f(n, m)$ and $\lim_{m\to\infty} \lim_{n\to\infty} f(n, m)$ are equal.

This completes the proof.

In the Hilbert space case, the situation is fairly simplified because of the existence of the self adjoint projection on $\mathbf{x}_T(\delta)$. ($\mathbf{x}_T(\delta)$ being closed as T is decomposable). Below we recover the following result of Wadhwa (1973).

Theorem 2.2 — Let T be a decomposable operator on a Hilbert space H. Then T is normal plus a commuting quasi-nilpotent if and only if

$$\sigma(T, P(\delta)\, x) \subseteq \sigma(T, x)$$

for each $x \in H$ and each closed set δ in C, where $P(\delta)$ is the self adjoint projection with range $\mathbf{x}_T(\delta)$.

PROOF : We shall only show the sufficiency as necessity is obvious. Consider the family $P(\delta)$ of self adjoint projections which is bounded. So, to apply Theorem 2.1, we have only to show that

$$R(P'(\delta)) = \cap \{R(P'(\bar{G})) \mid G \supseteq \delta \quad \text{and } G \text{ is open}\}.$$

Now $P'(\bar{G}) = J_H P(\bar{G}))\, J_H^{-1}$, where J_H is the isometry from H onto H^* given by $(J_H x)\,(y) = (y, x)$ for all $x, y \in H$. (Refer Goldberg 1966, p. 74).

Hence $R(P'(\bar{G})) = J_H R(P(\bar{G}))$. Thus

$$\cap \left\{ R(P'(\bar{G})) \,\middle|\, G \supseteq \delta \text{ and } G \text{ is open} \right\}$$

$$= \cap \left\{ J_H \left(R(P(\bar{G})) \right) \,\middle|\, G \supseteq \delta \text{ and } G \text{ is open} \right\}$$

$$= J_H \left[\cap \{ R(P(\bar{G})) \mid G \supseteq \delta \text{ and } G \text{ is open} \} \right]$$

(as J_H is injective)

$$= J_H \left[\cap \{ \mathbf{x}_T(\bar{G}) \mid G \supseteq \delta \text{ and } G \text{ is open} \} \right]$$

$$= J_H(\mathbf{x}_T(\delta))$$

(as T is decomposable).

Also $\quad R(P'(\delta)) = J_H R(P(\delta))$

(as $P'(\delta) = J_H P(\delta) J_H^{-1}$)

$$= J_H(\mathbf{x}_T(\delta))$$

Thus

$$R(P'(\delta)) = \cap \{ R(P'(\bar{G})) \mid G \supseteq \delta \text{ and } G \text{ is open} \}.$$

Hence by Theorem 2.1, we get that T is spectral. By Wadhwa (1973) T has property $[B']$ introduced by Stampli (1966). Hence T must be of the type normal plus a commuting quasi-nilpotent.

Corollary 2.1 — Suppose $T \in B(H)$ and T is decomposable. Then T is spectral if and only if there exists a family $\{E(\delta) \mid \delta \text{ closed in } C\}$ of projections (not necessarily self adjoint) with $R(E(\delta)) = \mathbf{x}_T(\delta)$, $\sigma(T, E(\delta)\, x) \subseteq \sigma(T, x)$ for each $x \in H$ and δ closed in C and $\{BE(\delta)\, B^{-1} \mid \delta \text{ closed in } C\}$ is a family of self adjoint operators for some invertible operator B in $B(H)$.

PROOF : Necessity of the condition follows as in the proof of Theorem 2.1 except for the existence of B such that $\{BE(\delta)\, B^{-1}\}$ is a family of self adjoint operators. This follows from Theorem 4 of Dunford and Schwartz (1971, p. 1947).

To show the sufficiency, define $T_1 = BTB^{-1}$. Then T is decomposable by Colojoara and Foias (1968). Now we use Ex. 40 of Dunford and Schwartz (1971, p. 2080) according to which, if $T = A^{-1}SA$, then

$$\sigma(T, x) = \sigma(S, Ax) \text{ for all } x \in H.$$

Thus for any closed set δ of C

$$\begin{aligned} \mathbf{x}_{T_1}(\delta) &= B\mathbf{x}_T(\delta) \\ &= BE(\delta)\, H \\ &= BE(\delta)\, B^{-1}H \end{aligned}$$

Also
$$\begin{aligned} \sigma(T_1, BE(\delta)B^{-1}x) &= \sigma(T, E(\delta)\, B^{-1}x) \\ &\subseteq \sigma(T, B^{-1}x) \\ &= \sigma(T_1, x) \end{aligned}$$

for each δ closed in C and $x \in H$.

Hence by Theorem 2.2, T must be of the type normal plus a commuting quasi-nilpotent. Thus T is spectral.

Corollary 2.2 — Suppose T is decomposable operator on a Hilbert space. Then T is self adjoint plus a commuting quasi-nilpotent if and only if

$$\mathbf{x}_T(\delta) = \mathbf{x}_{T^*}(\delta) \quad \text{for each closed set } \delta.$$

Proof : Observe that if T is specral and S is scalar part of T then $\mathbf{x}_T(\delta) = \mathbf{x}_S(\delta)$ by Theorem 4 of Dunford and Schwartz (1971, p. 1934). This proves the necessary part.

If $\mathbf{x}_T(\delta) = \mathbf{x}_{T^*}(\delta)$, then $\mathbf{x}_T(\delta)$ is reducing subspace for T as $T^*\mathbf{x}_{T^*}(\delta) \subseteq \mathbf{x}_{T^*}(\delta)$ and so the self adjoint projection $P(\delta)$ on $\mathbf{x}_T(\delta)$ will commute with T which implies that $\sigma(T, P(\delta)\, x) \subseteq \sigma(T, x)$ for each $x \in H$ and each closed set δ. Thus by Theorem 2.2 T is normal plus a commuting quasi-nilpotent. Hence T^* is spectral. So by Theorem 2.2 of Colojoara and Foias (1968, p. 41), T is quasi-nilpotent equivalent to T^*. If S is scalar part of T then again by Colojoara and Foias (1968) $S = S^*$ and the proof is complete.

Corollary 2.3 — If T is a decomposable operator on a Hilbert space H, then T is unitary plus a commuting quasi-nilpotent if and only if T is invertible and

$$\mathbf{x}_{T^{-1}}(\delta) = \mathbf{x}_{T^*}(\delta).$$

Proof : Proof of the necessity part is straightforward. We prove the sufficiency part.

T^{-1} is decomposable by Corollary 1.11 of Colojoara and Foias (1968, p. 37). If

$\mathbf{x}_{T^{-1}}(\delta) = \mathbf{x}_{T^*}(\delta)$, then $\mathbf{x}_{T^{-1}}(\delta)$ is a reducing subspace for T^{-1}. Hence T^{-1} is normal plus a commuting quasi-nilpotent as in Corollary 2.2 Thus T^{-1} and T^* both are spectral and $\mathbf{x}_{T^{-1}}(\delta) = \mathbf{x}_{T^*}(\delta)$ implies that they are quasi-nilpotent equivalent. If scalar part of T^* is S then as in Corollary 2.2 quasi-nilpotent equivalence of T^{-1} and T^* implies that $S = (S^*)^{-1}$ and so S^* the scalar part of T is unitary.

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References

Colojoara, I., and Foias, C. (1966). Quasi-nilpotent equivalence of not necessarily commuting operators. *J. Math. Mech.*, **15**, 521–40.

———— (1968). Generalized Spectral Operators. Gordon and Breach, New York.

Dunford, N. (1954). Spectral operators. *Pacific J. Math.*, **4**, 321–54.

Dunford, N., and Schwartz, J. T. (1957). Linear Operators I. John Wiley and Sons, Inc., New York.

———— (1971). Linear Operators III. John Wiley and Sons, Inc., New York.

Foias, C. (1968). Spectral capacities and decomposable operators. *Rev. Roumaine Math. Pures Appl.*, **13**, 1539–45.

Goldberg, S. (1966). Unbounded Linear Operators, Theory and Applications McGraw-Hill Book Company, Inc., New York.

Munroe, M. E. (1959). Introduction to Measure and Integration. Addison Wesley Publ. Co., Inc., Massachusetts.

Stampli, J. G. (1966). Analytic extensions and spectral localization. *J. Math. Mech.*, **16**, 287–96.

Wadhwa, B. L. (1973). Decomposable and spectral operators on a Hilbert space. *Proc. Am. math. Soc.*, **40**, 112–14.