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ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Let $f(z) = z - \sum_2^\infty |a_n| z^n$ be analytic in the unit disk $E = \{z : |z| < 1\}$ and let $F(z) = (1 - \lambda) f(z) + \lambda z f'(z)$ for z in E where $\lambda \geqslant 0$. The paper deals with the mapping properties of $F(z)$ when $f(z)$ is known. For example, if $f(z)$ is univalent in E, then the disk in which $F(z)$ is always starlike of order δ, $0 \leqslant \delta < 1$, is determined. All results are sharp.

§1. Let S denote the class of functions of the form $f(z) = z + \sum_2^\infty a_n z^n$ that are analytic and univalent in the disk $E = \{z : |z| < 1\}$. A function $f(z) \in S$ is said to be starlike of order α, $0 \leqslant \alpha < 1$, denoted by $S^*(\alpha)$, if $\mathrm{Re}\,(zf'(z)/f(z)) > \alpha$ for z in E and is said to be convex of order α, $0 \leqslant \alpha < 1$, denoted by $K(\alpha)$, if $\mathrm{Re}\,(1 + (zf''(z)/f'(z)) > \alpha$ for z in E.

Let T be the functions in S of the form $f(z) = z - \sum_2^\infty |a_n| z^n$ and set $T^*(\alpha) = S^*(\alpha) \cap T$ and $K^*(\alpha) = K(\alpha) \cap T$. Also, let $P^*(\alpha)$ denote the class of functions $f(z) = z - \sum_2^\infty |a_n| z_n$ analytic in E satisfying $\mathrm{Re}\, f'(z) > \alpha$ for z in E. It is known (Silverman 1975) that if $f(z) \in T$, then

$$\sum_2^\infty n\,|a_n| \leqslant 1 \qquad \text{...(1.1)}$$

and a necessary and sufficient condition for $f(z)$ to be in $T^*(\alpha)$ is that

$$\sum_2^\infty \left(\frac{n - \alpha}{1 - \alpha}\right) |a_n| \leqslant 1. \qquad \text{...(1.2)}$$

It is also known (Sarangi and Uralegaddi 1978) that a necessary and sufficient condition for $f(z)$ to be in $P^*(\alpha)$ is that

$$\sum_{2}^{\infty} \left(\frac{n}{1-\alpha}\right) | a_n | \leqslant 1. \qquad ...(1.3)$$

Recently, Sarangi and Uralegaddi (1979) have studied the mapping properties of the function $F(z)$ defined by

$$2F(z) = (zf(z))', \qquad ...(1.4)$$

when $f(z)$ is in $T^*(\alpha)$, $K^*(\alpha)$ or $P^*(\alpha)$.

The purpose of this paper is to study the mapping properties of the function $F(z)$ defined by

$$F(z) = (1 - \lambda) f(z) + \lambda z f'(z), \ (\lambda \geqslant 0) \text{ for } z \text{ in } E \qquad ...(1.5)$$

when $f(z)$ is in T, $T^*(\alpha)$, $K^*(\alpha)$ or $P^*(\alpha)$.

For the case $\lambda = \frac{1}{2}$, (1.5) reduces to (1.4).

§2. *Theorem* 1 — Let $f(z)$ be a function in T and $F(z) = (1 - \lambda) f(z) + \lambda z f'(z)$ for z in E where $\lambda \geqslant 0$. Then $\text{Re}\,(zF'(z)/F(z)) > \delta$, $0 \leqslant \delta < 1$, for $| z | < r(\lambda, \delta)$, where

$$r(\lambda, \delta) = \inf_{n} \left(\frac{n(1-\delta)}{(n-\delta)(1-\lambda+n\lambda)} \right)^{1/(n-1)} \quad (n = 2, 3, 4 \ldots).$$

The result is sharp.

PROOF : We have $F(z) = (1 - \lambda) f(z) + \lambda z f'(z)$

$$= z - \sum_{2}^{\infty} (1 - \lambda + n\lambda) | a_n | z^n.$$

It suffices to show that the values for $zF'(z)/F(z)$ lie in a circle centered at 1 whose radius is $1 - \delta$ for $| z | < r(\lambda, \delta)$. We have

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{\sum_{2}^{\infty} (n-1)(1-\lambda+n\lambda) | a_n | z^{n-1}}{1 - \sum_{2}^{\infty} (1-\lambda+n\lambda) | a_n | z^{n-1}} \right|$$

$$\leqslant \frac{\sum_{2}^{\infty} (n-1)(1-\lambda+n\lambda) | a_n | \, | z |^{n-1}}{1 - \sum_{2}^{\infty} (1-\lambda+n\lambda) | a_n | \, | z |^{n-1}}.$$

The last expression is bounded above by $1 - \delta$ if

$$\sum_{2}^{\infty} (n-1)(1-\lambda+n\lambda)\ |a_n|\ |z|^{n-1}$$

$$\leqslant (1-\delta)\ \left(1 - \sum_{2}^{\infty} (1-\lambda+n\lambda)\,|a_n|\ |z|^{n-1}\right),$$

which is equivalent to

$$\sum_{2}^{\infty} \frac{(n-\delta)(1-\lambda+n\lambda)}{1-\delta}\ |a_n|\ |z|^{n-1} \leqslant 1. \qquad \ldots(2.1)$$

Since $f(z)$ is in T we have (1.1). Hence (2.1) will be true if

$$\frac{(n-\delta)(1-\lambda+n\lambda)}{1-\delta}\ |a_n|\ |z|^{n-1} \leqslant n\,|a_n| \quad (n = 2, 3, 4, \ldots). \qquad \ldots(2.2)$$

Solving (2.2) for $|z|$, we obtain

$$|z| \leqslant \left(\frac{n(1-\delta)}{(n-\delta)(1-\lambda+n\lambda)}\right)^{1/(n-1)} \quad (n = 2, 3, 4, \ldots). \qquad \ldots(2.3)$$

Setting $|z| = r(\lambda, \delta)$ in (2.3), the result follows.

The result is sharp, with the extremal function $f(z) = z - (z^n/n)$.

Theorem 2 — Let $f(z)$ be a function in $T^*(\alpha)$ and $F(z) = (1-\lambda) f(z) + \lambda z f'(z)$ for z in E, where $\lambda \geqslant 0$. Then $F(z)$ is starlike of order δ, $0 \leqslant \delta < 1$, for $|z| < r(\lambda, \delta, \alpha)$, where

$$r(\lambda, \delta, \alpha) = \inf_n \left(\frac{(n-\alpha)(1-\delta)}{(n-\delta)(1-\alpha)(1-\lambda+n\lambda)}\right)^{1/(n-1)}$$

$$(n = 2, 3, 4, \ldots).$$

The result is sharp.

Proof : The proof is similar to that of Theorem 1. The only difference is that the estimate (1.2), is used in place of (1.1).

The result is sharp for the function $f(z) = z - \dfrac{1-\alpha}{n-\alpha} z^n$.

Corollary 2.1 — Let $f(z) \in T^*(\alpha)$ then $f(z)$ is starlike of order δ, $0 \leqslant \delta < 1$, in $|z| < r(0, \delta, \alpha) = \inf_n \left(\dfrac{(n-\alpha)(1-\delta)}{(n-\delta)(1-\alpha)}\right)^{1/(n-1)}$ for $n = 2, 3, 4, \ldots$.

The result is sharp.

Corollary 2.2 — Let $f(z) \in T^*(\alpha)$, then $f(z)$ is convex of order δ, $0 \leqslant \delta < 1$, in $|z| < r(1, \delta, \alpha) = \inf_n \left(\frac{(n-\alpha)(1-\delta)}{n(n-\delta)(1-\alpha)} \right)^{1/(n-1)}$ for $n = 2, 3, 4, \ldots$. The result is sharp.

Corollary 2.3 — Let $f(z) \in T^*(\alpha)$ and $c \geqslant 0$, then $F(z) = \frac{(z^c f(z))'}{(1+c)\, z^{c-1}}$ for z in E is starlike of order δ, $0 \leqslant \delta < 1$, in $|z| < r\left(\frac{1}{1+c}, \delta, \alpha\right)$, where

$$r\left(\frac{1}{1+c}, \delta, \alpha\right) = \inf_n \left(\frac{(1+c)(1-\delta)(n-\alpha)}{(n+c)(1-\alpha)(n-\delta)} \right)^{1/(n-1)}$$

for $n = 2, 3, 4, \ldots$.

This result is sharp.

Theorem 3 — Let $f(z) \in K^*(\alpha)$ and $F(z) = (1-\lambda) f(z) + \lambda z f'(z)$ for z in E, where $\lambda \geqslant 0$. Then $F(z)$ is close-to-convex in E if $\lambda < 1/(1-\alpha)$ and $F(z)$ is convex of order δ, $0 \leqslant \delta < 1$, in $|z| < r(\lambda, \delta, \alpha)$, where $r(\lambda, \delta, \alpha)$ is as stated in Theorem 2. The result is sharp.

PROOF: We have $F'(z) = (1-\lambda) f'(z) + \lambda (z f'(z))'$.

Therefore $\quad \text{Re}\, \frac{F'(z)}{f'(z)} = (1-\lambda) + \lambda\, \text{Re}\left(1 + \frac{z f''(z)}{f'(z)} \right) \geqslant 1 - \lambda + \lambda\alpha,$

since $f(z) \in K^*(\alpha)$. $1 - \lambda + \lambda\alpha > 0$ if $\lambda < 1/(1-\alpha)$. Hence $F(z)$ is close-to-convex in E if $\lambda < 1/(1-\alpha)$. We now prove that $F(z)$ is convex of order δ, $0 \leqslant \delta < 1$, in $|z| < r(\lambda, \delta, \alpha)$.

We have $\quad zF'(z) = (1-\lambda)\, z f'(z) + \lambda z (z f'(z))'$ for z in E. ...(2.4)

Since $f(z) \in K^*(\alpha)$, it follows that $zf'(z) \in T^*(\alpha)$. So, applying, Theorem 2 with $zf'(z)$ in place of $f(z)$, it follows from (2.4) that $zF'(z)$ is starlike of order δ in $|z| < r(\lambda, \delta, \alpha)$ or equivalently, that $F(z)$ is convex of order δ in $|z| < r(\lambda, \delta, \alpha)$.

The result is sharp for the function $f(z) = z - \frac{(1-\alpha)}{n(n-\alpha)} z^n$.

Theorem 4 — Let $f(z)$ be a function in $P^*(\alpha)$ and $F(z) = (1-\lambda) f(z) + \lambda z f'(z)$ for z in E, where $\lambda \geqslant 0$. Then $\text{Re}\,(F'(z)) > \delta$ for $|z| < r(\lambda, \delta, \alpha)$, where

$$r(\lambda, \delta, \alpha) = \inf_n \left(\frac{(1-\delta)}{(1-\alpha)(1-\lambda+n\lambda)} \right)^{1/(n-1)} \quad \text{for } n = 2, 3, 4 \ldots$$

This result is sharp.

PROOF: It is sufficient to prove that the values for $F'(z)$ lie in a circle centered at 1 whose radius is $1 - \delta$ for $|z| < r(\lambda, \delta, \alpha)$

We have $| F'(z) - 1 | = | \sum_2^\infty n(1 - \lambda + n\lambda) | a_n | z^{n-1} |$.

Hence $| F'(z) - 1 | \leqslant 1 - \delta$ if

$$\sum_2^\infty n(1 - \lambda + n\lambda) \; | a_n | \; | z |^{n-1} \leqslant 1 - \delta. \quad \ldots(2.5)$$

Since $f(z)$ is in $P^*(\alpha)$, we have (1.3). The remaining part of the proof is similar to that of Theorem 1.

The estimate is sharp for the function $f(z) = z - \frac{1 - \alpha}{n} z^n$.

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