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ON NEW SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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For a given $\alpha \geqslant 0$, we denote by $L(\alpha)$ the class of functions $f(z) = z + a_2 z^2 + \ldots$ which are regular in the unit disc, $E = \{z: |z| < 1\}$ and satisfy the condition $\text{Re}\,(D^{\alpha} f(z)/z) > o$, $z \in E$, where $D^{\alpha} f(z) = (z/(1-z)^{\alpha+2}) * f(z)$ ($*$ denotes the Hadamard product or convolution of analytic functions). Among other things it is shown that $L(\alpha)$ is a subclass of S, the usual class of normalised univalent functions in E.

Let us denote by A the family of functions f which are regular in the unit disc, $E = \{z : |z| < 1\}$ and normalised such that $f(0) = 0, f'(0) = 1$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ are in A, then so is their Hadamard product or convolution denoted by $(f * g)(z)$, and defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. We denote by S, C and K the subclasses of A whose members are univalent, close-to-convex and convex in E, respectively. A function f in A is said to be starlike of order α, $\alpha \leqslant 1$, in E if and only if

$$\text{Re}\,\frac{zf'(z)}{f(z)} > \alpha \qquad (z \in E),$$

and we denote by $S^*(\alpha)$ the class of all such functions. It is known that $S^*(0) \subset S$ and that $\alpha_1 \leqslant \alpha_2$ implies that $S^*(\alpha_2) \subset S^*(\alpha_1)$.

We shall denote by P' the subclass of C consisting of functions f which satisfy the condition : $\text{Re}\, f'(z) > 0$, $z \in E$.

Following Ruscheweyh (1977) we say that a function $f \in A$ is pre-starlike of order $\alpha \leqslant 1$ if, and only if,

$$\text{Re}\ \frac{f(z)}{z} > \tfrac{1}{2}, \qquad \text{if}\ \alpha = 1\ (z \in E),$$

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S^*(\alpha), \ \text{if}\ \alpha < 1.$$

We shall denote by R_α the class of pre-starlike functions of orders $\alpha \leqslant 1$. It is readily seen that $R_{1/2} \equiv S^*(1/2)$ and $R_0 \equiv K$.

Suffridge (1976) proved the following (see also Ruscheweyh 1977).

Theorem A — For $\beta \leqslant \alpha \leqslant 1$, we have $R_\beta \subset R_\alpha$.

The de la Vallée Poussin mean of order n of a function $f(z) = \sum_{k=1}^{\infty} a_k z^k$, regular in E, is defined by

$$V_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$$

$$= \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \ldots + \frac{n(n-1)\ldots 2.1}{(n+1)(n+2)\ldots(2n)} a_n z^n.$$

It is known (Pólya and Schoenberg 1958) that for the function $k(z)=z(1-z)^{-1} \in K$ each $V_n(z, k)$ is convex in E.

Let $\alpha \geqslant 0$ be a given real number and denote by $L(\alpha)$ the subclass of A whose members f satisfy the condition

$$\text{Re}\ \frac{D^\alpha f(z)}{z} > 0 \qquad (z \in E)$$

where

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+2}} * f(z).$$

It follows immediately $f \in L(0)$ if and only if $\text{Re}\, f'(z) > 0$ in E and as such $L(0) = P' \subset C \subset S$.

We shall need the following result due to Ruscheweyh and Sheil-Small (1973).

Lemma 1 — Let F be regular in E and satisfy

$$\text{Re}\ F(z) > 0 \qquad (z \in E)$$

Then for every $f \in K$ and every $g \in S^*(0)$, we have

$$\text{Re } \frac{(f * g F)(z)}{(f * g)(z)} > 0 \qquad (z \in E).$$

In Theorem 2 below we prove that for each given $\alpha \geqslant 0$, $L(\alpha)$ is a subclass of the class of close-to-convex functions. This result will follow at once from the following more general one.

Theorem 1 — Supose $F(z)$ is regular in E, $F(0) = 1$ and $F(E)$ is a convex schlicht domain. Let $\alpha \geqslant 0$ be given and suppose that for $f \in A$, $D^{\alpha} f(z)/z$ is subordinate to $F(z)$ in E, or in symbols

$$\frac{D^{\alpha} f(z)}{z} \prec F(z) \qquad \ldots(1)$$

in E. Then for every β, $-2 \leqslant \beta \leqslant \alpha$, we have

$$\frac{D^{\beta} f(z)}{z} \prec F(z) \qquad \ldots(2)$$

in E.

PROOF : Let $\alpha \geqslant 0$ be given. If $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, then

$$D^{\alpha} f(z) = \frac{z}{(1-z)^{2+\alpha}} * f(z)$$

$$= z + (2+\alpha)\, a_2 z^2 + \frac{(2+\alpha)(3+\alpha)}{2!}\, a_3 z^3 + \ldots$$

Define ψ by

$$\psi(z) = z + \frac{(2+\beta)}{(2+\alpha)}\, z^2 + \frac{(2+\beta)(3+\beta)}{(2+\alpha)(3+\alpha)}\, z^3 + \ldots .$$

Clearly ψ belongs to A and

$$\frac{z}{(1-z)^{2-2(-\alpha/2)}} * \psi(z) = z + (2+\beta)\, z^2 + \frac{(2+\beta)(3+\beta)}{2\,!}\, z^3 + \ldots$$

$$= \frac{z}{(1-z)^{2+\beta}} = q(z), \text{ say.} \qquad \ldots(3)$$

For all $\beta \geqslant -2$, the function $q(z)$ is readily seen to be in $S^*(-\beta/2)$ and hence in $S^*(-\alpha/2)$ if $\beta \leqslant \alpha$. Thus from (3) we conclude that ψ is pre-starlike of order $(-\alpha/2) \leqslant 0$, provided $\beta \leqslant \alpha$. Since $R_0 \equiv K$, in view of Theorem A it follows that ψ belongs to K and consequently we may write

$$\frac{\psi(z)}{z} = \int_0^{2\pi} \frac{1}{1 - ze^{it}}\, d\mu(t)$$

where μ is a probability measure on $[0, 2\pi]$.

Now suppose that the condition (1) is satisfied. Then in view of the definition of ψ, we have

$$\frac{D^{\beta}f(z)}{z} = \frac{D^{\alpha}f(z) * \psi(z)}{z} = \frac{D^{\alpha}f(z)}{z} * \frac{\psi(z)}{z}$$

$$= \frac{D^{\alpha}f(z)}{z} * \int_0^{2\pi} \frac{1}{(1 - ze^{it})} \, d\mu(t)$$

$$= \int_0^{2\pi} \frac{D^{\alpha}f(ze^{it})}{ze^{it}} \, d\mu(t)$$

$$\prec F(z), \qquad z \in E.$$

This completes the proof of Theorem 1.

For $\alpha = 0$, $\beta = -1$ and $F(z) = (1 + z)/(1 - z)$, Theorem 1 yields the known result:

Corollary — If $f \in A$ and $\text{Re}\, f'(z) > 0$, $z \in E$, then $\text{Re}\, f(z)/z > 0$, in E.

Theorem 2 — $L(\alpha) \subseteq L(\beta)$ for all β, $0 \leqslant \beta \leqslant \alpha$, and hence $f \in L(\alpha)$, $\alpha \geqslant 0$, implies that f is close-to-convex in E.

PROOF : Taking $F(z) = (1 + z)/(1 - z)$ in Theorem 1, we conclude that $f \in L(\alpha)$ implies that $f \in L(\beta)$ for all β, $0 \leqslant \beta \leqslant \alpha$. Since $L(0) = P'$, the theorem follows.

Theorem 3 — Let $\alpha \geqslant 0$ and $f \in L(\alpha)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Denote by $V_n(z, f)$ and $s_n(z, f)$ respectively the de la Vallée Poussin mean of order n and the nth partial sum of $f(z)$. Then for each n, $n = 1, 2, 3, \ldots$, we have

(a) $$\text{Re}\, \frac{D^{\alpha} V_n(z, f)}{z} > 0 \quad (z \in E)$$

and

(b) $$\text{Re}\, \frac{D^{\alpha} s_n(z, f)}{z} > 0$$

in the disc $|z| < r_n$, where r_n is the largest value of r, $0 < r < 1$, such that

$$\min_{\theta} \text{Re} \left\{ \frac{1 - r^n e^{ni\theta}}{1 - re^{i\theta}} \right\} > \tfrac{1}{2} \qquad \ldots(4)$$

for all r, $r \leqslant r_n$.

PROOF : Since $f \in L(\alpha)$, the function $F(z) = (D^\alpha f(z)/z)$ is regular and satisfies $\operatorname{Re} F(z) > 0$ in E. Also the function $g(z) = z$ belongs to $S^*(0)$. Hence in view of Lemma 1, we infer that for every $\varphi \in K$,

$$\operatorname{Re} \frac{\varphi(z) * D^\alpha f(z)}{z} = \operatorname{Re} \frac{\varphi(z) * (D^\alpha f(z)/z) \,.\, z}{\varphi(z) * z} > 0, \ (z \in E). \qquad \ldots(5)$$

Now
$$\frac{D^\alpha(\varphi * f)(z)}{z} = \frac{1}{z}\left[\frac{z}{(1-z)^{2+\alpha}} * (\varphi * f)(z)\right]$$
$$= \frac{1}{z}\left[\varphi(z) * \left\{\frac{z}{(1-z)^{2+\alpha}} * f(z)\right\}\right]$$
$$= \frac{\varphi(z) * D^\alpha f(z)}{z}. \qquad \ldots(6)$$

From (5) and (6) we have

$$\operatorname{Re} \frac{D^\alpha(\varphi * f)(z)}{z} > 0 \qquad (z \in E),$$

showing that

$$\varphi * f \in L(\alpha) \quad \text{for all } \varphi \in K. \qquad \ldots(7)$$

Now, as remarked earlier, the de la Vallée Poussin mean of order n, $V_n(z, k)$, of the function $k(z) = z(1-z)^{-1}$ is convex in E. Taking $\varphi(z) = V_n(z, k)$ in (7), we deduce that

$$V_n(z, f) = V_n(z, k) * f(z) \in L(\alpha).$$

This completes the proof of (a).

To prove (b), put

$$\sigma_n(z) = z + z^2 + \ldots + z^n = \frac{z(1-z^n)}{1-z}.$$

Then
$$s_n(z, f) = z + \sum_{k=2}^{n} a_k z^k = f(z) * \sigma_n(z)$$

and hence

$$\frac{D^\alpha s_n(z, f)}{z} = \frac{D^\alpha(f(z) * \sigma_n(z))}{z} = \frac{D^\alpha f(z)}{z} * \frac{\sigma_n(z)}{z}. \qquad \ldots(8)$$

Now it is known (Nehari and Netanyahu 1957) that if $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ are regular in E, $p(0) = q(0) = 1$ and $\operatorname{Re} p(z) > 0$, $\operatorname{Re} q(z) > \frac{1}{2}$ in E, then $\operatorname{Re}((p * q)(z)) > 0$, $z \in E$. (This result also follows from the proof of Theorem 1). Since by hypothesis $\operatorname{Re}(D^\alpha f(z)/z) > 0$, $z \in E$, the desired result follows immediately from (8) and the definition of $\sigma_n(z)$.

To prove that the number r_n cannot be replaced by any larger one, consider the function f_0 defined by

$$f_0(z) = z + \frac{2}{(\alpha + 2)} z^2 + \frac{2.2!}{(\alpha + 2)(\alpha + 3)} z^3 + \dots .$$

clearly $f_0 \in A$ and

$$\text{Re} \frac{D^\alpha f_0(z)}{z} = \text{Re}\left(\frac{1+z}{1-z} \right) > 0 \qquad (z \in E)$$

showing that in fact $f_0 \in L(\alpha)$.

Now,
$$\frac{D^\alpha s_n(z, f_0)}{z} = \left\{ \frac{D^\alpha f_0(z)}{z} * \frac{\sigma_n(z)}{z} \right\}$$

$$= \left\{ \frac{1+z}{1-z} * \frac{1-z^n}{1-z} \right\} = (1 + 2z + 2z^2 + \dots + 2z^{n-1})$$

$$= \frac{1 + z - 2z^n}{1-z} = 2\left\{ \frac{1-z^n}{1-z} - \tfrac{1}{2} \right\}.$$

Therefore, $\text{Re} \dfrac{D^\alpha s_n(z, f_0)}{z} > 0$ if $\text{Re} \left[\dfrac{1-z^n}{1-z} \right] > \tfrac{1}{2}$.

This completes the proof of Theorem 3.

Letting $\alpha = 0$ in Theorem 3, we obtain the following new results pertaining to the class P'.

Corollary 1 — The de la Vallée Poussin mean $V_n(z, f)$, $n \geqslant 1$, of every function $f \in P'$ satisfies the condition $\text{Re}\, V_n'(z, f) > 0$, $z \in E$, and is therefore close-to-convex (and hence univalent) in E.

Corollary 2 — The nth partial sum $s_n(z, f)$, $n \geqslant 2$, of each function $f \in P'$ satisfies the condition $\text{Re}\, s_n'(z, f) > 0$ in $|z| < r_n$ and is therefore close-to-convex (and hence univalent) in $|z| < r_n$.

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