*Indian J. pure appl. Math.*, 12(6) : 749–752, June 1981

# FUNCTIONS WITH UNIVALENT DERIVATIVES AND OF IRREGULAR GROWTH

MOHAMMAD SALMASSI

*Department of Mathematics, University of Tehran, Tehran, Iran*

AND

S. M. SHAH

*Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506, U.S.A.*

(*Received 23 August* 1980)

Let $E$ denote the class of functions $f$ analytic in the unit disc $D$, normalized so that $f(0) = 0 = f'(0) - 1$ and such that each $f^{(k)}$, $k > 0$, is univalent in $D$. It is shown here that there exist functions $f$ of given lower order $\lambda$ less than the order, and type not exceeding log 2.

## 1. INTRODUCTION

If $f \in E$ then it is known that (Shah and Trimble 1969) that $f$ is a transcendental entire function of exponential type less than $2\sqrt{3}$. The known examples of functions in $E$, with type $\pi$ are (i) $f(z) = (e^{\pi z} - 1)/\pi$ and (ii) $f(z) = \{G(z) - G(0)\}/G'(0)$ where $G(z) = e^{\pi z} + a_1 z + a_2 z^2$ and $a_1$, $a_2$ are suitable positive constants (Lachance 1980). These functions are of perfectly regular growth $(1, \pi)$, that is $\lim\limits_{r\to\infty} \dfrac{\log M(r)}{r} = \pi$. There are no known examples of functions in $E$ with type $> \pi$. Shah and Trimble (1971) have constructed functions of irregular growth but these functions are of zero type. In this paper we construct a function $f$ in $E$ of irregular growth and positive type.

*Theorem* — Given three numbers $\lambda$, $\rho$, $T$ such that $0 \leqslant \lambda < \rho \leqslant 1$, $0 \leqslant T \leqslant \log 2$, there exists a function $f$ belonging to $E$ and of lower order $\lambda$, order $\rho$ and type $T$.

## 2. PROOF

Since the examples given by Shah and Trimble (1971) cover the case $0 \leqslant \lambda < \rho \leqslant 1$, $T = 0$ we need consider when

$$0 \leqslant \lambda < 1,\ \rho = 1,\ 0 < T \leqslant \log 2. \qquad \ldots(2.1)$$

(a) Suppose first $\lambda > 0$. Let $\{m_k\}_1^\infty$ be a rapidly increasing sequence of positive integers, $m_1 > (2/T)^{1/(1-\lambda)}$, $N_k = (m_k)!$, $k = 1, 2, \ldots$; and let $\{x_k\}_{k=1}^\infty$ be a sequence of positive integers. Suppose further that

$$m_{k+1} > (N_k + x_k),\ k = 1, 2, \ldots . \qquad \ldots(2.2)$$

We define below another sequence $\{b_n\}$ and for the convenience of notation we write sometimes $b_n = b(n)$. Let $b_1 = 1$, $b_n = T$, $1 < n \leqslant m_1$,

$$b_n = T,\ m_k \leqslant n < N_k,\ k = 1, 2, \ldots,$$

$$b_n = \exp\,[[1 - \lambda^{-1}] \log n],\ n = N_k,\ k = 1, 2, \ldots,$$

$$b(N_k + j) = b(N_k)\,[1 + jN_k^{-1}],\ 1 \leqslant j \leqslant x_k,\ k = 1, 2, \ldots,$$

where $x_k$ is the largest integer such that $b(N_k + x_k) \leqslant T$. This choice of $x_k$ is possible since $b(N_k)\,[1 + N_k^{-1}] \leqslant T$ and $\Pi_1^\infty\,[1 + n^{-1}]$ is divergent to $+\infty$;

$$b(N_k + x_k + j) = T, \quad j = 1, 2, \ldots, m_{k+1} - N_k - x_k - 1,$$

$$k = 1, 2, \ldots .$$

Hence we have,

$$\frac{b(n+1)}{b(n)} \leqslant 1 + \frac{1}{n},\ n \geqslant 1; \qquad \ldots(2.3)$$

and $\quad b(n) \leqslant T,\ n \geqslant 2.$

Further for $n \geqslant 0$,

$$\sum_{k=1}^{\infty} \frac{1}{k!} \prod_{j=n}^{n+k-1} b_{j+2} \leqslant \sum_{k=1}^{\infty} \frac{T^k}{k!} = e^T - 1 \leqslant 1. \qquad \ldots(2.4)$$

Let the function $f$ be defined by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{b_1 b_2 \ldots b_n}{n!} e^{i\alpha_n} z^n,\ \alpha_n \in \mathbb{R}. \qquad \ldots(2.5)$$

Then for $|\,z\,| = r$,

$$\left.\begin{aligned} |\,f(z)\,| &\leqslant r + \sum_{n=2}^{\infty} \frac{b_1 b_2 \ldots b_n}{n!} r^n \\ &\leqslant r + \sum_{n=2}^{\infty} \frac{T^{n-1} r^n}{n!} = \frac{e^{rT} - 1}{T}. \end{aligned}\right\} \qquad \ldots(2.6)$$

Hence $f$ is an entire function of exponential type $\leqslant T$. Further

$$\sum_{n=2}^{\infty} n \frac{|\,f^{(n)}(0)\,|}{n!} = \sum_{n=2}^{\infty} \frac{b_1 b_2 \ldots b_n}{(n-1)!}$$

$$\leqslant \sum_{n=2}^{\infty} \frac{T^{n-1}}{(n-1)!} = e^T - 1 \leqslant 1$$

and consequently $f$ is starlike univalent in $D$. A similar argument shows that each $f^{(k)}$, $k = 1, 2, \ldots$, is univalent in $D$ (cf. Buckholtz and Shah 1980, 1981) Hence $f \in E$.

By (2.3)

$$\left\{\left|\frac{f^{(n+1)}(0)}{(n+1)\, f^{(n)}(0)}\right|\right\}_1^\infty$$

is a non-increasing sequence of positive numbers, and so (Shah 1946) writing $r_n = n/b_n$, we have

$$\text{the lower order of } f = \liminf_{n\to\infty} \frac{\log n}{\log r_n}$$

$$= \left\{1 + \limsup_{n\to\infty}\left[\frac{-\log b_n}{\log n}\right]\right\}^{-1} = \lambda;$$

and

$$\text{the order of } f = \limsup_{n\to\infty} \frac{\log n}{\log r_n}$$

$$= \left\{1 + \liminf_{n\to\infty}\left[\frac{-\log b_n}{\log n}\right]\right\}^{-1} = 1.$$

Further (Boas 1954), since the order is 1, the type is given by

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \frac{1}{e} \limsup_{n\to\infty} n\left[\frac{b_1 b_2 \ldots b_n}{n!}\right]^{1/n}.$$

When $n = N_{k+1} - 1$,

$$n\left[\frac{b_1 b_2 \ldots b_n}{n!}\right]^{1/n} = \exp\left\{\log n + \frac{1}{n}\sum_{k=1}^{n} \log b_k - \frac{1}{n}(n \log n - n + O(1))\right\}$$

$$= e(1 + o(1)) \exp\left\{\frac{1}{n}\sum_{j=1}^{N_k + x_k} \log b_j\right\}$$

$$\times\ T^{(N_{k+1} - N_k - x_k - 1)/(N_{k+1} - 1)}.$$

Now

$$\left|\frac{\log b_1 b_2 \ldots b(N_k + x_k)}{N_{k+1} - 1}\right| \leqslant \frac{|\log b_1| + \ldots + |\log b(N_k + x_k)|}{N_{k+1} - 1}$$

$$\leqslant \frac{(N_k + x_k)}{N_{k+1} - 1}\left\{\left[\frac{1}{\lambda} - 1\right] \log N_k\right\}$$

$$= O\left[\frac{m_{k+1} \log N_k}{N_{k+1}}\right] = o(1). \qquad \ldots(2.7)$$

Hence we conclude from (2.6) that the exponential type of $f$ is $T$.

(b) Now suppose $\lambda = 0$. Let $\phi(x)$ be a strictly increasing positive function on $(1, \infty)$ and let $b_n = n^{-\phi(n)}$ for $n = N_k$, $k = 1, 2, \ldots$ . Choose $m_1$ so large that $\phi(N_1) \log N_1 \geqslant \log (2/T)$. Then the corresponding function $f$ defined by (2.5) has type $T$ and lower order zero.

This completes the proof.

## 3. Remarks

(i) It follows from our choice of $x_k$ that when $\lambda > 0$, $x_k \sim TN_k^{1/\lambda}$. If we choose $m_1$ sufficiently large, $m_1 > m_0(\lambda, T)$, $N_k = (m_k)!$, and $m_{k+1} \geqslant [(T+1)\, N_k^{1/\lambda}]$, $k = 1, 2, \ldots$ then the conditions on the sequences $\{m_k\}$ and $\{x_k\}$ are satisfied and (2.7) holds. When $\lambda = 0$, we may take $\phi(n) = \log n$, $N_k = (m_k)!$

$$m_{k+1} \geqslant [(T+1)\, N_k^{1+\phi(N_k)}], \quad k = 1, 2, \ldots$$

then also the conditions on these sequences are satisfied.

(ii) Ted Suffridge has shown (unpublished) that if $G(z) = e^{az} + ce^{-az}$, $0 < a \leqslant \pi$, $|\, c \,| \leqslant e^{-2a}$, then $f(z) = \{G(z) - G(0)\}/G'(0)$ belongs to the class $E$. This function $f$ is of perfectly regular growth not exceeding $(1, \pi)$.

(iii) In his doctoral dissertation the first author (Salmassi 1978) constructed a function $f$ belonging to the class $E$ and of irregular growth $(1, (\log 2)/e)$.

(iv) If $f$ is defined by (2.5), where $\{\alpha_n\}$ is any sequence of real numbers, and $\{b_n\}$ is any sequence of positive members such that $b_1 = 1$ and (2.4) with $T = \log 2$ is satisfied, then $f$ is in class $E$ and of growth not exceeding $(1, \log 2)$.

## References

Boas, R. P. (1954). Entire Functions. Academic Press, New York, p. 11.

Buckholtz, J. D., and Shah, S. M. (1980). Absolute starlike and absolute convex functions. *J. d'Analyse Math.*, **38**, 113–43.

——— (1981). Analytic functions with constraints on coefficients. *J. Nonlinear Anal.*, (to appear).

Lachance, M. (1980). Remark on functions with all derivatives univalent. *Int. J. Math. Sci.*, **3**, 193–96.

Shah, S. M. (1946). On the lower order of integral functions. *Bull. Am. math. Soc.*, **52**, 1046–52.

Shah, S. M., and Trimble, S. Y. (1969). Univalent functions with univalent derivatives. *Bull. Am. math. Soc.*, **75**, 153–57.

——— (1971). Entire functions with univalent derivatives. *J. Math. Anal. Applic.*, **33**, 220–29.