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ON FINITE STRAIN THEORY FOR ELASTIC-INELASTIC BODIES

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This paper is devoted to propose a different approach on finite strain theory for elastic-inelastic bodies. It has been established that the rate of thermoelastic strain is independent of the inelastic spin in general. Constitutive relations have been obtained from appropriate constitutive equations and the laws of thermodynamics. Flow rules have been discussed in brief for the inelastic strain.

1. INTRODUCTION

The finite strain theory of elastic-inelastic bodies has been of long interest. An excellent account of the same can be found in recent papers (Lee 1969, Haddow and Hrudey 1971, Kratochvil and Dillon 1968, Kratochvil 1973, Green and Naghdi 1965, Capriz and Saha 1976). Haddow and Hrudey (1971) have arrived at the result that the rate of thermoelastic strain depends on plastic spin.

In the present paper, a different approach has been proposed and it has been established that the rate of thermoelastic strain does not depend, in general, on the inelastic spin. A stress free intermediate configuration, as in Lee (1969), Haddow and Hrudey (1971) and Sedov (1965), at initial temperature has also been considered in addition to the usual initial and final configurations to discuss the elastic-inelastic motion of the body. Constitutive relations have been derived from appropriate constitutive equations and the restrictions on them within the framework of thermodynamics as given in Coleman and Noll (1963). The flow rules have been discussed for the inelastic strain rate appearing in the final inequality. Some stability conditions have also been explained using the results of Coleman and Gurtin (1967) when inelastic strain can be expressed as an internal parameter.

2. DEFORMATIONS AND KINEMATICS

The elastic and inelastic deformations of the body are considered in its three states: initial configuration at time t_0, final configuration at time t and intermediate configuration. Intermediate configuration of the body is attained when there is no stress and the body is at initial temperature. Deformations are measured by continuous one-one mapping from initial to final configuration of the body which are expressed by the functional relations

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$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t); \mathbf{X} = \mathbf{X}(\mathbf{x}, t) \quad \text{...(2.1)}$$

where **X** and **x** represent the position of a typical particle of the body in initial and final configurations respectively. The gradient of the total deformation is given by $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$. Then we must have

$$\det \mathbf{F} \neq 0 \quad \text{...(2.2)}$$

since the deformation is continuous and one-one i.e., $d\mathbf{x} = 0$ if and only if $d\mathbf{X} = 0$.

Intermediate configuration of the body may also be called as an inelastically strained configuration. The distribution of the residual inelastic strain does not satisfy, in general, the compatibility conditions and thus the space corresponding to it is non-Euclidean. Concept of deformation issues from an intermediate configuration leads to a decomposition

$$\mathbf{F} = \mathbf{N}\mathbf{M} \quad \text{...(2.3)}$$

where **N** and **M** are non-singular and represent the gradients of thermoelastic and inelastic parts of deformations. A decomposition given by (2.3) is always possible whether the intermediate configuration is Euclidean or non-Euclidean.

The total, inelastic, elastic and thermoelastic strain tensors are respectively given by

$$\left.\begin{aligned} \mathbf{E} &= \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \ \mathbf{E}^{(M)} = \tfrac{1}{2}(\mathbf{M}^T\mathbf{M} - \mathbf{I}) \\ \mathbf{e}^{(e)} &= \tfrac{1}{2}(\mathbf{N}^T\mathbf{N} - \mathbf{I}), \ \mathbf{E}^{(N)} = \mathbf{M}^T\mathbf{e}^{(e)}\mathbf{M}. \end{aligned}\right\} \quad \text{...(2.4)}$$

From (2.4) it follows that

$$\mathbf{E} = \mathbf{E}^{(N)} + \mathbf{E}^{(M)}. \quad \text{...(2.5)}$$

The rate of total and inelastic deformations are respectively expressed as

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + L^T), \ \mathbf{D}^{(M)} = \tfrac{1}{2}(\mathbf{L}^{(M)} + \mathbf{L}^{(M)^T}) \quad \text{...(2.6)}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, $\mathbf{L}^{(M)} = \dot{\mathbf{M}}\mathbf{M}^{-1}$.

The dot above the letter denotes the differentiation with respect to time t.

The rates of total and inelastic strain tensors follow the relationships:

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F}, \ \dot{\mathbf{E}}^{(M)} = \mathbf{M}^T\mathbf{D}^{(M)}\mathbf{M}. \quad \text{...(2.7)}$$

Then, using (2.5) we obtain

$$\dot{\mathbf{E}}^{(N)} = \mathbf{M}^T\mathbf{D}^{(N)}\mathbf{M} \quad \text{...(2.8)}$$

where $\mathbf{D}^{(N)} = \mathbf{N}^T\mathbf{D}\mathbf{N} - \mathbf{D}^{(M)}$.

It is clear from (2.8) that the rate of thermoelastic strain, $\dot{\mathbf{E}}^{(N)}$, depends only on $\mathbf{D}$ and $\mathbf{D}^{(M)}$ and not on the inelastic spin, i.e., on $\frac{1}{2}(\mathbf{L}^{(M)} - \mathbf{L}^{(M)T})$, and $\dot{\mathbf{E}}^{(N)} = \mathbf{0}$, if $\mathbf{D} = \mathbf{0}$ and $\mathbf{D}^{(M)} = \mathbf{0}$. So the present approach overcomes the complications experienced in Haddow and Hrudey (1971).

In the special case, if the body is inelastically incompressible then $\det \mathbf{M} = 1$ and $\det \mathbf{F} = \det \mathbf{N} = \rho_0/\rho$, where ρ_0 and ρ are densities in the initial and final configurations respectively. In this case, we have $tr\ (\mathbf{D}^{(M)}) = 0$.

We obtain the following results

$$\mathbf{D} = (\mathbf{N}^{-1})^T (\mathbf{D}^{(M)} + \mathbf{D}^{(N)}) \mathbf{N}^{-1} = \tilde{\mathbf{D}}^{(M)} + \tilde{\mathbf{D}}^{(N)} \qquad ...(2.9)$$

where $\tilde{\mathbf{D}}^{(M)} = (\mathbf{N}^{-1})^T \mathbf{D}^{(M)}\mathbf{N}^{-1}$, $\tilde{\mathbf{D}}^{(N)} = (\mathbf{N}^{-1})^T \mathbf{D}^{(N)}\mathbf{N}^{-1}$.

This result agrees with that of Kratochvil (1973).

The total power $\mathbf{TD}$ can be written as

$$\mathbf{TD} = \mathbf{T}\tilde{\mathbf{D}}^{(N)} + \mathbf{T}\tilde{\mathbf{D}}^{(M)} \qquad ...(2.10)$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{T}\tilde{\mathbf{D}}^{(N)}$ and $\mathbf{T}\tilde{\mathbf{D}}^{(M)}$ are thermoelastic and inelastic powers respectively.

3. Constitutive Equations

We consider a thermodynamic process as in Coleman and Noll (1963) with one more field, the inelastic deformation gradient $\mathbf{M}$, and consider the following constitutive equations:

$$\left.\begin{array}{ll} \text{(a)} & \psi = \hat{\psi}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}) \\ \text{(b)} & \eta = \hat{\eta}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}) \\ \text{(c)} & \mathbf{Y} = \mathbf{Y}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}) \\ \text{(d)} & \tilde{\mathbf{q}} = \hat{\tilde{\mathbf{q}}}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}) \end{array}\right\} \qquad ...(3.1)$$

where ψ is the free-energy function, $\psi = \epsilon - \eta\theta$; $\mathbf{Y} = (\det \mathbf{F})\ \mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^T$ is the Piola-Kirchhoff stress tensor, $\mathbf{G} = \mathbf{F}^T \operatorname{grad} \theta$, $\tilde{\mathbf{q}} = \mathbf{F}^T\mathbf{q}$ and $\mathbf{q}$ is the heat flux vector. The motivation behind the consideration of such constitutive equations is quite clear; we want to write these in objective form*.

These equations are compatible with balance of linear momentum given by

$$\operatorname{div} (\mathbf{FY}) + \rho(\mathbf{b} - \ddot{\mathbf{x}}) = \mathbf{0}, \qquad ...(3.2)$$

*Under objectivity we consider similar changes for $\mathbf{M}$ as it is for $\mathbf{F}$ i.e., under a change of frame characterized by the orthogonal tensor $\mathbf{Q}$, we have $\mathbf{F} \rightarrow \mathbf{QF}$ and $\mathbf{M} \rightarrow \mathbf{QM}$ (Sidoroff 1973).

and the balance of energy given by

$$\rho\dot{\epsilon} = tr\,(\mathbf{Y}\dot{\mathbf{E}}) - (\rho_0/\rho)\,\text{div}\,\mathbf{q} + \rho r. \qquad ...(3.3)$$

Then, we write down the Clausius-Duhem inequality, which imposes restrictions on the constitutive equations, in the simplified form as

$$(\dot{\psi} + \dot{\theta}\eta) - \frac{1}{\rho_0}\,tr\,(\mathbf{Y}\dot{\mathbf{E}}) - \frac{1}{\rho\theta}\,\mathbf{q}\cdot\text{grad}\,\theta \leqslant 0. \qquad ...(3.4)$$

Then from (3.1) and (3.4), we obtain

$$\left(\frac{\partial\hat{\psi}}{\partial\theta} + \eta\right)\dot{\theta} + tr\left\{\left(\frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(N)}} - \frac{1}{\rho_0}\mathbf{Y}\right)\dot{\mathbf{E}}^{(N)}\right\} + tr\left(\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\dot{\mathbf{G}}\right)$$
$$+\, tr\left\{\left(\frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(M)}} - \frac{1}{\rho_0}\mathbf{Y}\right)\dot{\mathbf{E}}^{(M)}\right\} - \frac{1}{\rho\theta}\,\mathbf{q}\cdot\text{grad}\,\theta \leqslant 0. \qquad ...(3.5)$$

We make the assumption that the constitutive equations (3.1) are such that $\dot{\mathbf{E}}^{(N)}$, $\dot{\theta}$ and $\dot{\mathbf{G}}$ can be chosen arbitrarily at any material point $\tilde{\mathbf{X}}$ and at any instant of time $\tilde{t}$; the choice of $\dot{\mathbf{E}}^{(M)}$ on $\tilde{\mathbf{X}}$ and $\tilde{t}$ in a neighbourhood of $(\tilde{\mathbf{X}}, \tilde{t})$ will not influence the choice of $\mathbf{x}$ and t. From (3.5) we obtain

$$\text{(a)}\quad \eta = -\frac{\partial\hat{\psi}}{\partial\theta}, \quad \text{(b)}\quad \mathbf{Y} = \rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(N)}}, \quad \text{(c)}\quad \frac{\partial\hat{\psi}}{\partial\mathbf{G}} = \mathbf{0} \qquad ...(3.6)$$

and the residual inequality

$$tr\left\{\left(\frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(N)}} - \frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(M)}}\right)\dot{\mathbf{E}}^{(M)}\right\} + \frac{1}{\rho\theta}\,\mathbf{q}\cdot\text{grad}\,\theta \geqslant 0. \qquad ...(3.7)$$

The relation (3.6c) implies that ψ, and hence η and $\mathbf{Y}$ are independent of the temperature gradient. In other words, we may say that the temperature gradient can influence the heat flux only.

If $\dot{\mathbf{E}}^{(M)}$ can also be chosen arbitrarily, then (3.7) leads to the result

$$\frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(N)}} = \frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(M)}}. \qquad ...(3.8)$$

This means that ψ is a function of $\mathbf{E}^{(N)} + \mathbf{E}^{(M)} = \mathbf{E}$ i.e., $\hat{\psi} = f(\mathbf{E})$, which again implies that ψ is a function of $\mathbf{F}$ only. So, we conclude that the constitutive equations (3.1) depend on $\mathbf{F}$ only when $\dot{\mathbf{E}}^{(M)}$ is chosen arbitrarily and thus the decomposition $\mathbf{F} = \mathbf{NM}$ is meaningless. Such a decomposition can influence eqn. (3.1d) only.

If $\dot{\mathbf{E}}^{(M)}$ is not wholly arbitrary but depends on the choice of other variables [as in case of viscoplasticity (Capriz and Saha 1976, Valanis 1971)] then there exists a function $\mathbf{A}$ such that

$$\dot{\mathbf{E}}^{(M)} = \mathbf{A}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}). \qquad \ldots(3.9)$$

In this case the inelastic strain $\mathbf{E}^{(M)}$ plays the role of internal parameter as explained in Kratochvil and Dillon (1968), Capriz and Saha (1976) and Biot (1954). This provides a clue to approximate physical alteration in the body with microstructure such as dislocation arrangements in the crystalline materials.

Perfectly plastic body is characterized by an admissible domain in the state space $(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G})$, where the plastic law is given according to the position and motion of this state space as given in Green and Naghdi (1965) and Sidoroff (1975).

4. Inelastic Strain as an Internal Parameter

As we explained in section 3, that for viscoplastic materials there exists a functional relation of the type (3.9) and for such a case the strain $\mathbf{E}^{(M)}$ behaves like an internal parameter. Thus, we have the relationships:

$$\psi = \hat{\psi}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta),\ \eta = -\frac{\partial\hat{\psi}}{\partial\theta},\ \mathbf{Y} = \rho_0 \frac{\partial\hat{\psi}}{\partial\mathbf{E}^{(N)}},$$

$$\tilde{\mathbf{q}} = \hat{\tilde{\mathbf{q}}}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}) \text{ and } \dot{\mathbf{E}}^{(M)} = \mathbf{A}(\mathbf{E}^{(N)}, \mathbf{E}^{(M)}, \theta, \mathbf{G}). \qquad \ldots(4.1)$$

Following Coleman and Gurtin (1967), we can define the triplet

$$\Sigma_* = (\mathbf{E}_*^{(N)}, \mathbf{E}_*^{(M)}, \theta_*)$$

as an equilibrium state when

$$\mathbf{A}(\mathbf{E}_*^{(N)}, \mathbf{E}_*^{(M)}, \theta_*, \mathbf{0}) = \mathbf{0} \qquad \ldots(4.2)$$

and the domain of attraction of Σ_* at $\mathbf{E}^{(N)}$ = constant, θ = constant, as the set $\Gamma(\Sigma_*)$ of all $\mathbf{E}^{(M)}$ such the solution

$$\mathbf{E}^{(M)} = \mathbf{E}^{(M)}(t)$$

of all initial value problems

$$\dot{\mathbf{E}}_*^{(M)} = \mathbf{A}(\mathbf{E}_*^{(N)}, \mathbf{E}_*^{(M)}, \theta_*, \mathbf{0});\ \mathbf{E}^{(M)}\,|_{t=t_0} = \mathbf{E}_0^{(M)}; \qquad \ldots(4.3)$$

exists for all $t > t_0$ and tends to $\mathbf{E}_*^{(M)}$ i.e., $\mathbf{E}^{(M)}(t) = \mathbf{E}_*^{(M)}$ as $t \to \infty$.

Σ_* is asymptotically stable at $\mathbf{E}^{(N)}$ = constant, θ = constant if $\Gamma(\Sigma_*)$ contains a neighbourhood of $\mathbf{E}_*^{(M)}$. As a consequence of asymptotic stability when $\mathbf{G} = \mathbf{0}$, we have

$$\hat{\psi}(\mathbf{E}_*^{(N)}, \mathbf{E}^{(M)}, \theta_*) \geqslant \hat{\psi}(\mathbf{E}_*^{(N)}, \mathbf{E}_*^{(M)}, \theta_*)$$

for all $\mathbf{E}^{(M)}$ in some neighbourhood of $\mathbf{E}_*^{(M)}$ and this means that

$$\frac{\partial \hat{\psi}}{\partial \mathbf{E}^{(M)}} = 0 \text{ in } \Sigma_*. \qquad \ldots(4.4)$$

This implies that ψ is independent of plastic strain for such an asymptotically stable equilibrium state and hence the free-energy is a constant function only. This result can be summarized in what follows:

Theorem — The stress vanishes for an asymptotically stable equilibrium state Σ_* at constant elastic strain and at constant temperature on a particle with $\mathbf{G} = \mathbf{0}$.

This gives the idea of stress relaxation. In other words this theorem states that, under the above assumption $\mathbf{E}^{(M)}$ varies in such a way that the stress $\mathbf{Y}$ tends to zero as $t \to \infty$ when the total strain $\mathbf{E}^{(N)}$ is kept constant.

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