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STABILITY OF HYDROMAGNETIC STRATIFIED SHEAR FLOW

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The instability of the flow of an inviscid, incompressible, hydromagnetic fluid of variable density between two parallel planes in the presence of a magnetic field in the direction of the flow is investigated for the two cases: (i) weak magnetic field and (ii) linear velocity profile. In both the cases it is shown that the complex wave velocity of unstable modes lie in ellipse-type region whose one diameter coincides with the diameter of Howard's semi-circle, while the other diameter depends on magnetic force number and the Richardson number. The bounds of their growth rates are also shown to depend on them.

Kochar and Jain (1979) have shown that for any unstable mode the complex wave velocity c for hydrodynamic stratified shear flow must lie in a semi-ellipse type region whose major diameter coincides with the diameter of Howard's semi-circle (Howard 1961), while the minor diameter depends on stratification. Here we have investigated the hydromagnetic flow of an inviscid, incompressible electrically conducting fluid of variable density between two parallel rigid planes when there is an applied magnetic field in the direction of the flow. Two cases are discussed (i) weak magnetic field and (ii) linear velocity profile. It is shown that in both the cases the complex wave velocity of unstable modes lie in ellipse-type region whose one diameter coincides with the diameter of Howard's semi-circle while the other diameter depends on magnetic force number and the Richardson number. The bounds of their growth rate are shown to depend on them.

The linear instability of inviscid, incompressible heterogeneous hydromagnetic fluid flow between two parallel planes $y = y_1$ and y_2, is governed by the equation (Agarwal and Agrawal 1969),

$$D(\rho W^2 DF) - \rho k^2 W^2 F + \rho\beta g F - \rho S(D^2 - k^2) F = 0 \qquad \ldots(1)$$

where $W = U - c$, $\rho(y)$ is the density, $U(y)$ the velocity, S the magnetic force number, $\beta = -(D\rho)/\rho$ and c the complex wave velocity, with the boundary conditions: $F = 0$ at $y = y_1$ and y_2.

If (1) is multiplied by F^*, the complex conjugate of F, and integrated over (y_1, y_2) then real and imaginary parts of the resultant equation after some manuplations give that $a \leqslant c_r \leqslant b$ and

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 + S - \tfrac{1}{4}(a-b)^2\right]\int \rho Q + \int \rho\beta g \mid F \mid^2 \leqslant 0 \quad \ldots(2)$$

where $Q = \mid F' \mid^2 + k^2 \mid F \mid^2$ and $a \leqslant U(y) \leqslant b$.

For $\beta > 0$, the above inequality gives the semi-circle theorem that the complex wave velocity of the unstable modes will lie in the semi-circle given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 + S_{\min} - \tfrac{1}{4}(a-b)^2 \leqslant 0. \quad \ldots(3)$$

For weak magnetic field ($S \ll 1$) and large k we can neglect in (1) the term SD^2F as compared to Sk^2F. Further, taking the transformation $G = W^{1/2}F$, we have

$$(\rho WG')' - \rho k^2 WG - \rho(\tfrac{1}{2}L + X^2 \mid W \mid^{-1})\, U'^2 G = 0 \quad \ldots(4)$$

with boundary conditions : $G = 0$ at y_1 and y_2. In this equation

$$L = (\rho U')'/\rho U'^2,\quad X^2 = \tfrac{1}{4} - J - S(k^2/U'^2) \quad \text{and} \quad J = \beta g/U'^2.$$

Multiplying eqn. (4) with G^*, the complex conjugate of G, integrating the resultant equation over (y_1, y_2) and separating the real and imaginary parts, we have

$$\int \rho(U - c_r)\, Q_0 + \int \rho\, [\tfrac{1}{2}L + (U - c_r)\, X^2 \mid W \mid^{-2}]\, U'^2 \mid G \mid^2 = 0 \quad \ldots(5)$$

$$c_i\, [\int \rho Q_0 - \int \rho \mid W \mid^{-2} X^2 U'^2 \mid G \mid^2] = 0 \quad \ldots(6)$$

where $Q_0 = \mid G' \mid^2 + k^2 \mid G \mid^2$.

For $c_i > 0$,

$$\int \rho Q_0 = \int \rho X^2 U'^2 \mid W \mid^{-2} \mid G \mid^2. \quad \ldots(7)$$

Multiplying eqn. (7) by c_r and adding it in (5), we have

$$\int \rho U Q_0 + \int [\tfrac{1}{2}L + X^2 \mid W \mid^{-2}(U - 2c_r)]\, \rho U'^2 \mid G \mid^2 = 0.$$

This gives $a \int \rho Q_0 + \int [\tfrac{1}{2}L + X^2 \mid W \mid^{-2}(U - 2c_r)]\, \rho U'^2 \mid G \mid^2 \leqslant 0$

and $b \int \rho Q_0 + \int [\tfrac{1}{2}L + X^2 \mid W \mid^{-2}(U - 2c_r)]\, U'^2 \mid G \mid^2 \geqslant 0.$

Using (7) in these relations, we have

$$\int [\tfrac{1}{2}L + X^2 \mid W \mid^{-2}(a + U - 2c_r)]\, \rho U'^2 \mid G \mid^2 \leqslant 0$$

and $\int [\tfrac{1}{2}L + X^2 \mid W \mid^{-2}(b + U - 2c_r)]\, \rho U'^2 \mid G \mid^2 \geqslant 0.$

These inequalities give

$$c_i^2 \leqslant \frac{X^2_{\min}}{U'^2_{\max}\, L_{\min}}\,(c_r - a) \quad \text{or} \quad c_i^2 \leqslant -\frac{X^2_{\min}\,(b - c_r)}{L_{\max} U'^2_{\max}} \quad \ldots(8)$$

according as $L_{\min} > 0$ or $L_{\max} < 0$.

Now $|G'|^2 = |D(W^{1/2}F)|^2$

$$\geqslant |W|\ |F'|^2 + \tfrac{1}{4}U'^2\,|W|^{-1}\,|F|^2 - |U'|\ |F|\ |F'|.$$

Let $A^2 = \int \rho\,|W|\,Q$, $B^2 = \int \rho U'^2\,|W|^{-1}\,|F|^2$ and $J_0 = \min(\beta g/U'^2)$. With these notations we have

$$\int \rho Q_0 \geqslant A^2 + \tfrac{1}{4}B^2 - \int \rho\,|U'|\ |F|\ |F'|.$$

Using Schwarz inequality we have

$$\int \rho\,|U'|\ |F|\ |F'| \leqslant \{\int \rho U'^2\,|W|^{-1}\,|F|^2\}^{1/2}\{\int \rho\,|W|\ |F'|^2\}^{1/2}$$
$$\leqslant \{\int \rho U'^2\,|W|^{-1}\,|F|^2\}^{1/2}\{\int \rho\,|W|\,Q\}^{1/2} \leqslant AB.$$

Thus

$$\int \rho Q_0 \geqslant A^2 + \tfrac{1}{4}B^2 - AB = (A - \tfrac{1}{2}B)^2. \qquad \ldots(9)$$

Moreover

$$\int \rho Q_0 = \int \rho X^2 U'^2\,|W|^{-1}\,|F|^2 \leqslant X_0^2 B^2 \qquad \ldots(10)$$

where $X_0^2 = \frac{1}{4} - J_{\min} - (S_{\min}k^2/U'^2_{\max})$. The eqns. (9) and (10) then give

$$(A - \tfrac{1}{2}B)^2 \leqslant X_0^2 B^2 \quad \text{or} \quad \tfrac{1}{2} - X_0 \leqslant \frac{A}{B} \leqslant \tfrac{1}{2} + X_0. \qquad \ldots(11)$$

Let $\lambda^2 = \min[\int \rho\,|G'|^2/\int \rho\,|G|^2]$. Then

$$\int \rho Q_0 = \int \rho\,|G'|^2 + k^2\int \rho\,|G|^2 \geqslant (\lambda^2 + k^2)\int \rho\,|G|^2$$

and so

$$\lambda^2 + k^2 \leqslant \frac{\int \rho Q_0}{\int \rho\,|G|^2} \leqslant \frac{X_0^2 B^2}{\int \rho\,|G|^2} \leqslant \frac{X_0^2 \int \rho U'^2 |W|^{-2} |G|^2}{\int \rho\,|G|^2} \leqslant \frac{X_0^2 U'^2_{\max}}{c_i^2}.$$

Hence

$$c_i^2 \leqslant \frac{X_0^2\,U^2_{\max}}{\lambda^2 + k^2} \leqslant X_0^2\,U'^2_{\max}/\lambda^2. \qquad \ldots(12)$$

Further, let $\nu^2(F) = \int \rho\,|F'|^2/\int \rho\,|F|^2$. So

$$\frac{\int \rho Q}{\int \rho |F|^2} = \nu^2 + k^2 \leqslant \nu^2 - \lambda^2 + \frac{X_0^2 U'^2_{\max}}{c_i^2} \leqslant \frac{\nu^2 X_0^2\,U'^2_{\max}}{(\lambda^2 + k^2)\,c_i^2}. \qquad \ldots(13)$$

Also we have

$$\nu^2 \leqslant \nu^2 + k^2 = \frac{\int \rho Q}{\int \rho\,|F|^2} \leqslant \frac{\int \rho\,|W|\,Q\,|W|^{-1}}{\int \rho\,|W|^{-1}U'^2\,|F|^2\,|W|\,U'^{-2}} \leqslant \frac{A^2 U'^2_{\max}}{B^2 c_i^2}.$$

Taking the upper bound of A/B from (11), we have

$$\nu^2 \leqslant \frac{U'^2_{\max}}{c_i^2}\left(\tfrac{1}{2} + X_0\right)^2.$$

Using this in relation (13) we get

$$\frac{\int \rho Q}{\int \rho \mid F \mid^2} \leqslant \frac{X_0^2\, U'^4_{\max}}{(\lambda^2 + k^2)\, c_i^4} \left(\tfrac{1}{2} + X_0\right)^2$$

or $$\int \rho \mid F \mid^2 \geqslant \frac{(\lambda^2 + k^2)\, c_i^4 \int \rho Q}{X_0^2\, U'^4_{\max} \left(\tfrac{1}{2} + X_0\right)^2} \geqslant \frac{(\lambda^2 + k^2)\, c_i^4 \int \rho Q}{X_0^2\, U'^4_{\max}} \geqslant \frac{\lambda^2 c_i^4 \int \rho Q}{X_0^2\, U'^4_{\max}}$$

Further, $$\int \rho(\beta q + Sk^2) \mid F \mid^2 \geqslant J_0 U'^2_{\max} \int \rho \mid F \mid^2 \geqslant \frac{J_0 c_i^4 \lambda^2}{X_0^2\, U'^2_{\max}} \int \rho Q.$$

Using this result in the last term of (2) for weak magnetic field we see that the complex velocity of the unstable modes shall lie inside a semi-ellipse type region given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2\left(1 + \frac{\lambda^2 J_0 c_i^2}{X_0^2\, U'^2_{\max}}\right) \leqslant \tfrac{1}{4}(a-b)^2. \qquad \ldots(15)$$

For potentially unstable flow ($\beta < 0$), the complex wave velocity of unstable modes lie inside a semi-ellipse type region given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2\left(1 - \frac{J_c \lambda^2 c_i^2}{X_1^2 U'^2_{\max}}\right) \leqslant \tfrac{1}{4}(a-b)^2, \qquad \ldots(16)$$

where $J_c = \max\left(\mid \beta \mid g/U'^2\right)$ and $X_1^2 = \tfrac{1}{4} - J_c - S(k^2/U'^2_{\max})$.

Finally, if the velocity is linear (with a and b as lower and upper bounds, respectively) then eqn. (1) with the transformation $G = W^{1/2}F$ gives, following the usual procedure,

$$c_i\left[\int \rho(1 + S \mid W \mid^{-2})\, Q_0 - \int \rho \mid W \mid^{-2} U'^2 Z^2 \mid G \mid^2\right] = 0,$$

where $$Z^2 = \tfrac{1}{4} - J + \tfrac{3}{4} S \mid W \mid^{-4} \{3(U - c_r)^2 - c_i^2\}.$$

For $c_i > 0$ we have $\int \rho(1 + S \mid W \mid^{-2})\, Q_0 = \int \rho \mid W \mid^{-2} U'^2 Z^2 \mid G \mid^2$.

We see that

$$Z^2 \leqslant \tfrac{1}{4} - J_0 + \max\left[\tfrac{3}{4} S\{3(U-c_r)^2 - c_i^2\} \mid W \mid^{-4}\right]$$

$$\leqslant \tfrac{1}{4} - J_0 + \tfrac{9}{4}\frac{S_{\max}}{c_i^2} - \frac{48}{25}\frac{S_{\min} c_i^2}{(b-a)^4} \equiv Z_0^2$$

Thus $\int \rho Q_0 \leqslant \int \rho(1 + S \mid W \mid^{-2})\, Q_0 \leqslant Z_0^2 \int \rho \mid W \mid^{-2} U'^2 \mid G \mid^2$

$$\leqslant Z_0^2 \int \rho U'^2 \mid W \mid^{-1} \mid F \mid^2 = Z_0^2 B^2.$$

Then following the analysis as in the case of weak magnetic field we see that an upper bound of growth rate of unstable modes, i.e. c_i^2, is $Z_0^2\, U'^2/\lambda^2$ and the complex wave velocity of unstable mode lie in a semi-ellipse type region

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2\left(1 + \frac{J_0 \lambda^2 c_i^2}{Z_0^2\, U'^2}\right) \leqslant \tfrac{1}{4}(a-b)^2 - S_{\min}. \qquad \ldots(17)$$

For pontentially unstable flow ($J < 0$) the complex wave velocity of unstable mode lie in semi-ellipse type region

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2\left(1 - \frac{\lambda^2 J_c c_i^2}{Z_1^2\, U'^2}\right) \leqslant \tfrac{1}{4}(a-b)^2 - S_{\min}. \qquad \ldots(18)$$

where $$Z_1^2 = \tfrac{1}{4} + J_c + \frac{9 S_{\max}}{4 c_i^2} - \frac{48 S_{\min} c_i^2}{25(b-a)^4}.$$

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