Indian J. pure appl. Math., 12(6) : 769–778, June 1981

EFFECTIVE INTERACTIONS FOR TRITON

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(*Received* 19 *June* 1980; *after revision* 29 *September* 1980)

The unitary-scheme model bases with symmetric-exchange force potentials are used to calculate the binding energy, the nuclear wave function and the charged nuclear radius of triton. Approximate method for calculating the Bartlett and the Heisenberg forces in terms of those of Wigner and Majorana has been applied. Two symmetric-exchange force potentials are suggested in view of the results obtained.

1. INTRODUCTION

As is well-known a central problem in nuclear physics is to understand the nature of the isolated nucleon-nucleon interactions and to explain the properties of complex nuclei in terms of these nuclear forces. The description of nuclear systems can be attempted in a fundamental or microscopic sense by explicitly accounting for the motion of each nuleon. This approach is generally quite complex and closed-form analytic expression is seldom available. Alternately, one might develop relevant macroscopic or many-body concepts, models, and parameters in terms of which a satisfactory treatment of complex nuclei could be sought. Moreover, the fact that the triton nucleus is the simplest many-body nuclear system after the deutron makes it attractive to apply the theoretical techniques available to study complex nuclei.

In the studies concerning this problem, one may distinguish between five basic approaches (Macharadze and Mikhelashvili 1971) : (1) the direct method of solving the Faddeev equation; (2) the K-harmonic method; (3) the shell model with oscillator potential; (4) the traditional variational methods; (5) and the translational invariant shell model, that means, the unitary sheme model (USM) of Vanagas (1971).

In previous work (Salem and Doma 1971, 1977; Doma and Youssef 1979) the present author studied the triton problem using mainly the techniques of the unclear shell model, modified nuclear shell model, and unitary sheme model, respectively. In these investigations use was made once of the central part of the Hu-Massey potential (Hu and Massey 1949), then of both the central and tensor forces of Massey potential, and lastly of the Gogny *et al.* interaction (Gogny *et al.* 1970). However, in the present work we tackle the triton problem differently in the sense

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that the USM is used together with effective interaction for the first time. Moreover the approximate method is carried out to evaluate the mean value of the Bartlett and the Heisenberg interactions in terms of those of Wigner's and Majorana's and then getting the average value of the central interparticle interaction of the exchange type forces.

Here, symmetric-exchange force potentials with Gaussian radial dependence have been used. The choice of the Gaussian form was made for two reasons : it makes calculations much easier as well as it leads to faster convergence. These potentials depend on two parameters, namely, the depth and range of the potentials, V_0 and r_c, respectively. The parameter V_0 is allowed to vary in the energy range $25 \leqslant V_0 \leqslant 30$ Mev and then the binding energy of ^3H is minimized with respect to the parameter r_c.

The ground-state wave function of ^3H is expanded in terms of bases of the USM with number of quanta of excitations $0 \leqslant N \leqslant 10$. The so-called two-particle orbital density matrices are introduced to simplify the calculations. The charged nuclear radius of ^3H is also calculated. Good results have been obtained and two symmetric-exchange force potentials are suggested.

2. Construction of the Energy Matrix

As usual, the inter-particle Hamiltonian of a system with A nucleons can be written in the form (Doma and Youssef 1979):

$$H = H^{(0)} + V' \qquad \ldots(2.1)$$

where

$$H^{(0)} = \frac{1}{A} \sum_{i<j=1}^{A} \left[\frac{1}{2m} (\mathbf{P}_i - \mathbf{P}_j)^2 + \tfrac{1}{2} m\omega^2 (\mathbf{r}_i - \mathbf{r}_j)^2 \right] \qquad \ldots(2.2)$$

is the USM Hamiltonian and V' is the residual interaction. Assuming the residual interaction to be central and of exchange type forces one can write

$$V' = H_p - \frac{m\omega^2}{2A} \sum_{i<j=1}^{A} (\mathbf{r}_i - \mathbf{r}_j)^2 \qquad \ldots(2.3)$$

where

$$H_p = \sum_{i<j=1}^{A} V(\,|\,\mathbf{r}_i - \mathbf{r}_j\,|\,)\ {}^{ts}X_{ij} \qquad \ldots(2.4)$$

The operator ${}^{ts}X_{ij}$ is given by (Vanagas 1971)

$${}^{ts}X_{ij} = C_W + (-1)^{s+t+1} C_M + (-1)^{s+1} C_B + (-1)^{t+1} C_H \qquad \ldots(2.5)$$

where C_W, C_M, C_B, and C_H are the Wigner, Majorana, Bartlett and Heisenberg constants, respectively, and s, t are the resultant spin, isospin momenta of the pair nucleons i and j.

The central potential $V(\,|\,\mathbf{r}_i - \mathbf{r}_j\,|\,)$ is assumed to be Gaussian:

$$V_{ij} \equiv V(\,|\,\mathbf{r}_i - \mathbf{r}_j\,|\,) = V_0 \exp(-r_{ij}^2/r_c^2). \qquad \ldots(2.6)$$

The method introduced by Vanagas (1971) represents the operator H_p as a sum of scalar products of orbital and spin-isospin irreducible operators each of rank, χ, which assumes the following three irreducible representations (IR) of the symmetric group S_A :

$$\chi = [A], [A-1, 1] \quad \text{and} \quad [A-2, 2].$$

Applying Wigner-Eckart theorem of the symmetric operators to these scalar products the mean value, E_p, of the operator H_p, given in eqn. (2.4) above, with respect to the USM bases, then, assumes the form

$$\begin{aligned} E_p &= C_W E_W + C_M E_M + C_B E_B + C_H E_H \\ &= \sum_{\chi} F_p^{(\chi)}(C_W, C_M, C_B, C_H) \langle\, [f] \,\|\, V^{(\chi)} \,\|\, [f'] \,\rangle \end{aligned} \qquad \ldots(2.7)$$

where

$$\begin{aligned} F^{(\chi)} = \delta_{[f_S]\,[f_T]}^{[f'_S]\,[f'_T]} [\delta_{[f]}^{[f']} (C_W + C_M \langle\, [f] \,\|\, P^{(\chi)} \,\|\, [f'] \,\rangle) \\ + C_B W^{(r\sigma)} \langle\, [f_S] \,\|\, S^{(\chi)} \,\|\, [f_S] \,\rangle + C_H W^{(rr)} \\ \times \langle\, [f_T] \,\|\, T^{(\chi)} \,\|\, [f_T] \,\rangle]. \end{aligned} \qquad \ldots(2.8)$$

In eqn. (2.7) $V^{(\chi)}$ is an irreducible operator constructed, arbitrarily, from the two particle operator V_{ij}, eqn. (2.6). The operators $P^{(\chi)}$, $S^{(\chi)}$ and $T^{(\chi)}$ are the orbital, spin and isospin-exchange operators of symmetry type χ, constructed from the original exchange operators by the usual manner. The two operators $W^{(r\sigma)}$ and $W^{(rr)}$ are factorized in terms of $6j$-symbols of the symmetric group S_A. The two IR $[f_S]$ and $[f_T]$, appearing in the right-hand side of eqn. (2.8), are given by

$$\left.\begin{aligned} [f_S] &= \left[\frac{A}{2} + S, \frac{A}{2} - S\right] \\ [f_T] &= \left[\frac{A}{2} + T, \frac{A}{2} - T\right] \end{aligned}\right\} \qquad \ldots(2.9)$$

where S and T are the spin and isospin momenta of the nucleus. $[f]$ is an IR of the symmetric groug S_A.

Setting $C_B = C_H = 0$ and using a square-well potential of parameters depending on the orbital angular momentum and the parity of the two-nucleon states approximate relations for the Bartlett and the Heisenberg energies in terms of those of Wigner's and Majorana's are obtained. Inserting these relations in eqn. (2.7) one then obtains

$$E_p = \left[(C_W + C_M) + (C_B - C_H)\frac{\Lambda_\sigma - \Lambda_\tau}{n + \Lambda}\right] E^{(+)} + \left[(C_W - C_M) + (C_B + C_H)\frac{\Lambda_\sigma + \Lambda_\tau}{n - \Lambda}\right] E^{(-)}. \quad \ldots(2.10)$$

Here

$$E^{(\pm)} = \tfrac{1}{2}(E_W \pm E_M), \quad \ldots(2.11)$$

are the interaction energies in orbital-symmetric and antisymmetric states respectively and $n = \frac{1}{2}A(A-1)$ is the number of nucleon pairs. In eqn. (2.10) Λ, Λ_σ, and Λ_τ are the values of the operator

$$\Lambda([\lambda]) = \sum_{i=1}^{A} \lambda_i(\lambda_i - 2i + 1);$$
$$[\lambda] = [\lambda_1 \ldots \lambda_i \ldots \lambda_A], \quad \ldots(2.12)$$

with $[\lambda] = [f]$, $[fs]$ and $[f_T]$ respectively.

The energy eigenvalues and the corresponding eigenfunctions of the USM Hamiltonian $H^{(0)}$, eqn. (2.2), are given by (Doma and Youssef 1979)

$$E_N^{(0)} = [N + \tfrac{3}{2}(A-1)]\,\hbar\omega \quad \ldots(2.13)$$

$$\psi_N^{(0)} \equiv |\, A\Gamma M_L;\, \Gamma_S M_S M_T \,\rangle \equiv |\, AN\{\rho\}^{\alpha LM_L}_{(\nu)[f]};\, SM_S TM_T \,\rangle \quad \ldots(2.14)$$

Basis functions with given total momentum J may be constructed from those of eqn. (2.14), in the usual manner, as follows

$$\left|A\Gamma SJM_J TM_T\right\rangle = \sum_{M_L + M_S = M_J} (LM_L, SM_S \mid JM_J)\left|A\Gamma M_L;\, \Gamma_S M_S M_T\right\rangle \quad \ldots(2.15)$$

where $(LM_L, SM_S \mid JM_J)$ is the Clebsh-Gordan coefficient of the rotational group R_3.

The nuclear wave function is expanded in terms of the USM basis functions as follows

$$\left|J^\pi TM_J M_T\right\rangle = \sum_{\Gamma, S} C^{J^\pi T}_{\Gamma, S}\left|A\Gamma SJM_J TM_T\right\rangle \quad \ldots(2.16)$$

where $C^{J^\pi T}_{\Gamma,S}$ are the state-expansion coefficients, π defines the parity of the state, and the number of quanta of excitation N is permitted to be either even or odd integer depending on the parity.

3. Method of Calculations

The mean values of the Wigner and the Majorana interactions have the forms (Vanagas 1971)

$$E_W = \sum_{\epsilon\epsilon' l} I_{\epsilon l,\epsilon' l} Q^{[f]L}_{\epsilon l,\epsilon' l} \qquad \ldots(3.1)$$

$$E_M = \sum_{\epsilon\epsilon' l} (-1)^{\epsilon'} I_{\epsilon l,\epsilon' l} Q^{[f]L}_{\epsilon l,\epsilon' l} \qquad \ldots(3.2)$$

where $I_{\epsilon l,\epsilon' l}$ are the radial integrals

$$I_{\epsilon l,\epsilon' l} = \int_0^\infty R_{\epsilon l}(r)\, R_{\epsilon' l}(r)\, V_0 e^{-r^2/r_c^2}\, r^2 dr \qquad \ldots(3.3)$$

with $r = r_{ij}$ and $R_{\epsilon l}(r)$ being the radial wave function of the two-nucleon state. In the right-hand side of eqn. (3.1) there appears the two-particle orbital density matrix $Q^{[f]L}$ which is factorized in terms of the two-particle orbital fractional parentage coefficients (Doma 1979) as follows

$$Q^{[f]L}_{\epsilon l,\epsilon' l'} = \frac{A(A-1)}{2} \sum_{\bar{\Gamma}} \frac{d_{[\bar{f}]}}{d_{[f]}} \langle A\Gamma \mid A - 2\bar{\Gamma};\, 2\,[f_a]\, \epsilon l \rangle$$

$$\times \langle A\Gamma' \mid A - 2\bar{\Gamma};\, 2\,[f'_a]\, \epsilon' l' \rangle \qquad \ldots(3.4)$$

where $d_{[f]}$ is the dimension of the IR $[f]$, and ϵ denoting $\epsilon = 2n + l$ in which n is the radial quantum number and l is the orbital angular momentum of the pair nucleons.

Once Wigner and Majorana energies are calculated the interaction energy matrix H_p with matrix elements E_p, [eqn. (2.7)] is obtained.

The oscillator operator appearing in eqn. (2.3) has matrix elements given by (Doma and Salem 1976)

$$\left\langle \psi^{(0)}_N \middle| \frac{m\omega^2}{2A} \sum_{i<j=1}^{A} r_{ij}^2 \middle| \psi^{(0)'}_N \right\rangle = \sum_{\epsilon\epsilon' l} K_{\epsilon l,\epsilon' l} Q^{[f]L}_{\epsilon l,\epsilon' l} \qquad \ldots(3.5)$$

where

$$K_{\epsilon l,\epsilon' l} = \frac{m\omega^2}{2A}\int_0^\infty R_{\epsilon l}(r)\, R_{\epsilon' l}(r)\, r^4\, dr = \frac{\hbar\omega}{A}\,[(\epsilon + \tfrac{3}{2})\,\delta^{\epsilon'}_{\epsilon}$$

$$- \tfrac{1}{2}\,(\epsilon - l + 2)^{1/2}\,(\epsilon + l + 3)^{1/2}\,\delta^{\epsilon'}_{\epsilon+2}$$

$$- \tfrac{1}{2}\,(\epsilon - l)^{1/2}\,(\epsilon + l + 1)^{1/2}\,\delta^{\epsilon'}_{\epsilon-2}]. \qquad \ldots(3.6)$$

Inserting certain values for the interaction constants C_W, C_M, C_B, and C_H, corresponding to symmetric forces, we have been able to diagonalize the energy matrices with respect to the model parameter $\hbar\omega$ and the potential parameters V_0 and r_c, which are allowed to vary in definite ranges in order to obtain best fit between the calculated binding energies and the corresponding experimental findings.

If we take into account the radius of the proton, r_p, then the charged nuclear radius $\mathcal{R}$ is given as usual by

$$\mathcal{R} = [\langle r_P^2 \rangle + \langle R_{NUC}^2 \rangle]^{1/2} \qquad \ldots(3.7)$$

where

$$R_{NUC}^2 = \frac{1}{A^2}\sum_{i<j=1}^{A} (\mathbf{r}_i - \mathbf{r}_j)^2. \qquad \ldots(3.8)$$

The matrix elements of R_{NUC}^2, and hence of $\mathcal{R}$, can be obtained directly by multiplying eqn. (3.5) by the factor $\dfrac{2}{m\omega^2 A}$.

4. Results and Conclusions

Basis functions of the USM, with number of quanta of excitations $0 \leqslant N \leqslant 10$, total orbital angular momentum $L = 0$, total spin $S = \frac{1}{2}$, total isotopic spin $T = \frac{1}{2}$, and orbital symmetry $[f] = [3]$ and $[21]$, are chosen in the calculations. Twenty of the two-particle orbital density matrices are constructed for the considered cases.

For our symmetric-exchange force potentials we considered the following two well-known sets of values for the constants C's of eqn. (2.5).

Case a

$C_W = 0.1333$, $C_M = -0.9333$, $C_B = -0.4667$, $C_H = -0.2667$ known as the Rosenfeld constants (Rosenfeld 1948).

Case b

$C_W = 0.1917$, $C_M = -1.0500$, $C_B = -0.5250$, $C_H = -0.3834$ taken in accordance with the Hu and Massey potential (1949).

For our basis functions the average value E_p, (2.10), assumes the form

$$E_p = [C_W + C_M]\, E^{(+)} + [(C_W - C_M) + \tfrac{4}{5}(C_B + C_H)\, \delta^{[f]}_{[3]}]\, E^{(-)} \quad ...(4.1)$$

Fixing V_0 and r_c and introducing a new parameter r_0, given by

$$r_0 = \sqrt{\frac{\omega m}{2\hbar}}\, r_c \quad ...(4.2)$$

the binding energy of triton is then minimized with respect to this parameter and the corresponding energy eigenfunctions are obtained. Using these wave functions the charged nuclear radius, $\mathcal{R}$, of triton is obtained as a function of the oscillator parameter $\hbar\omega$, which is obtained from the relation (4.2) corresponding to the value of r_0

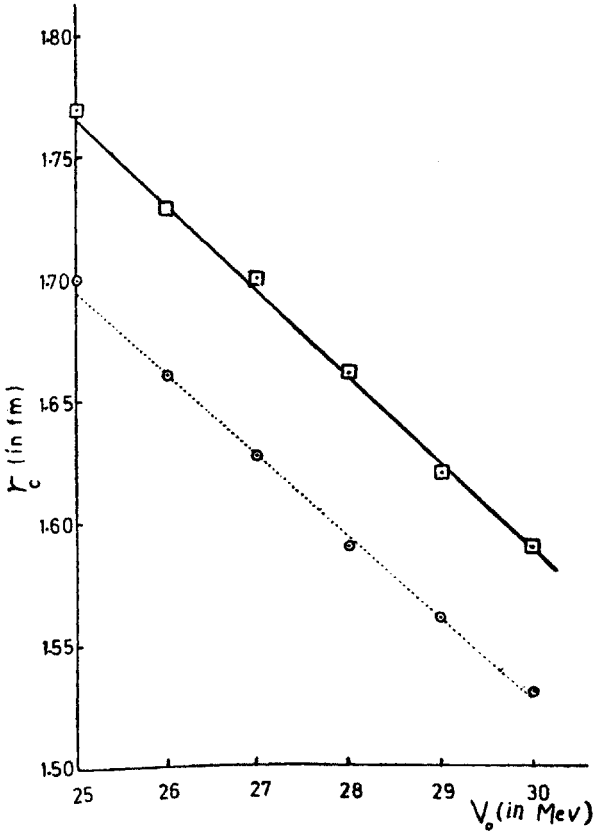

FIG. 1.

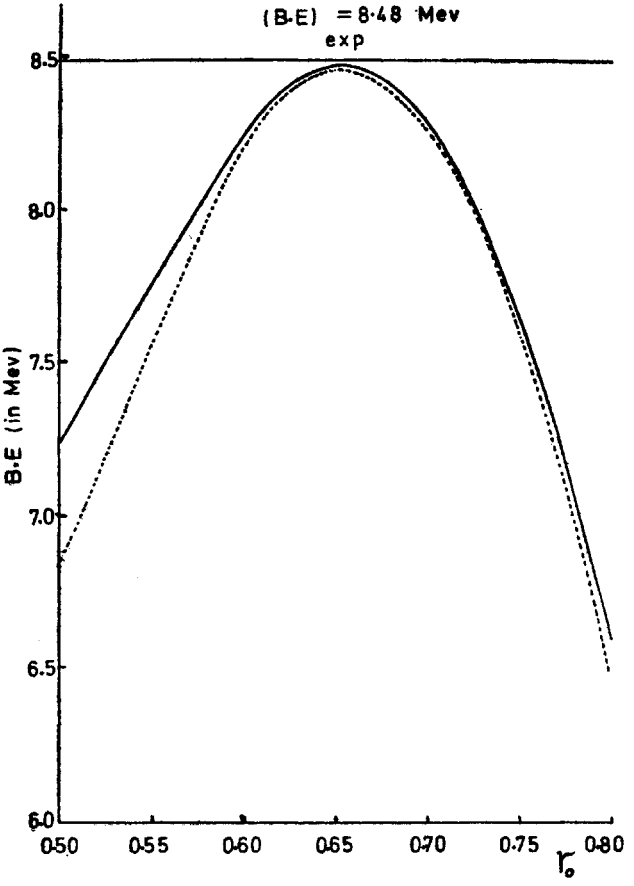

FIG. 2.

that gives good agreement between the calculated binding energy of triton and the corresponding experimental findings.

Variations of the binding energy and the charged nuclear radius of triton with the two parameters V_0 and r_c are estimated. In Fig. 1 the variation of the range parameter r_c with the depth parameter V_0 is presented. The solid curve corresponds to Case a and the dotted curve corresponds to Case b. Best fitting between the obtained results and the corresponding experimental findings for ^3H is obtained for the following two potentials:

Potential I

Case a with $r_c = 1.660$ fm, and $V_0 = 28$ Mev.

Potential II

Case b with $r_c = 1.626$ fm, and $V_0 = 27$ Mev.

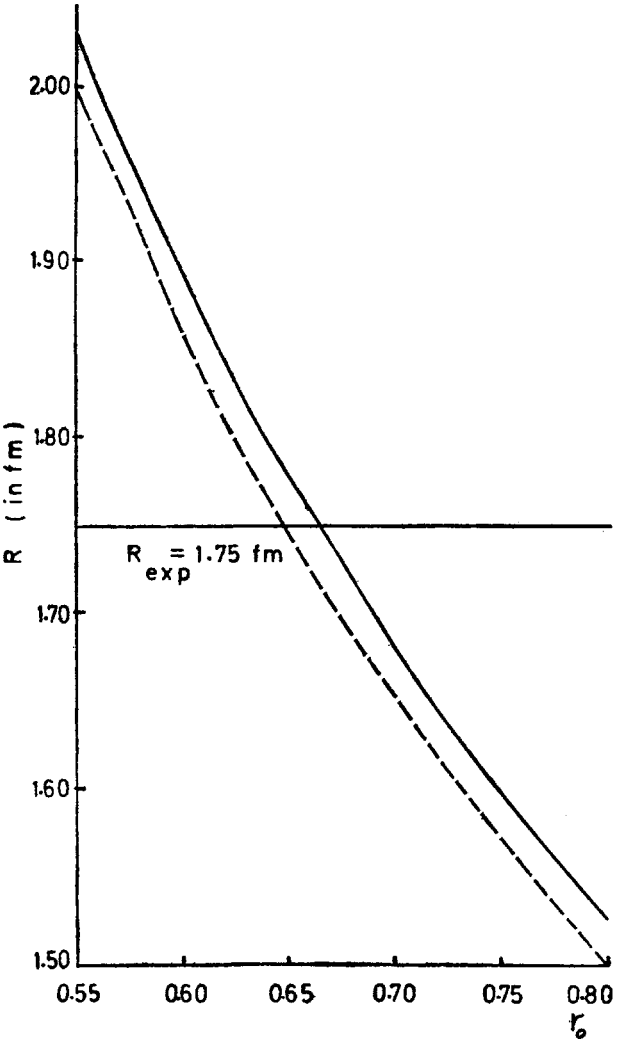

FIG. 3.

Figures 2 and 3 illustrate the dependence of the binding energy (B.E.) and the charged nuclear radius ($\mathcal{R}$) of triton upon the parameter r_0. The solid curve corresponds to Potential I and the dotted curve corresponds to Potential II. Best fitting occurs when $r_0 = 0.65$ for both cases giving $\hbar\omega = 12.777$, and 13.317 Mev for the two Potentials I and II, respectively.

For Potential I the obtained results for the binding energy, the charged nuclear radius, and the nuclear wave function of ^3H are

B.E. = 8.474 Mev, $\mathcal{R}$ = 1.777 fm, and

$$\Psi = 0.8095 \mid 0\ \{0\}\ (0)\ \ [3]\ 0\ \tfrac{1}{2}\ \rangle + 0.3081 \mid\ 2\ \ \{2\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.2890 \mid 2\ \{2\}\ (2)\ [21]\ 0\ \tfrac{1}{2}\ \rangle + 0.2811 \mid\ 4\ \ \{4\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.1841 \mid 4\ \{4\}\ (2)\ [21]\ 0\ \tfrac{1}{2}\ \rangle + 0.1451 \mid\ 6\ \ \{6\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.1038 \mid 8\ \{8\}\ (0)\ \ [3]\ 0\ \tfrac{1}{2}\ \rangle + 0.1467 \mid 10\ \{10\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$

For Potential II we get

B.E. = 8.461 Mev, $\mathcal{R}$ = 1.749 fm, and

$$\Psi = 0.8110 \mid 0\ \{0\}\ (0)\ \ [3]\ 0\ \tfrac{1}{2}\ \rangle + 0.3062 \mid\ 2\ \ \{2\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.2880 \mid 2\ \{2\}\ (2)\ [21]\ 0\ \tfrac{1}{2}\ \rangle + 0.2813 \mid\ 4\ \ \{4\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.1834 \mid 4\ \{4\}\ (2)\ [21]\ 0\ \tfrac{1}{2}\ \rangle + 0.1440 \mid\ 6\ \ \{6\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$
$$+ 0.1036 \mid 8\ \{8\}\ (0)\ \ [3]\ 0\ \tfrac{1}{2}\ \rangle + 0.1461 \mid 10\ \{10\}\ (0)\ [3]\ 0\ \tfrac{1}{2}\ \rangle$$

The two potentials give good agreement with the well-known experimental findings, concerning the binding energy and the charged nuclear radius of ^3H.

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