Indian J. pure appl. Math., 12(6) : 786–797, June 1981

PERFECT FLUID MODELS OF ANISOTROPIC UNIVERSE

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(*Received* 22 *February* 1980)

The general Bianchi type I cosmological solutions of Einstein field equations, corresponding to spatially homogeneous and anisotropic models containing barotropic fluid have been obtained and their properties have been discussed.

1. INTRODUCTION

In recent years, the effects of neutrino viscosity in primordial fireball (Doroskhevich *et al.* 1967, Misner 1967) have stimulated theoretical interest in anisotropic cosmological models and a large number of exact solutions have been obtained (Heckmann and Schücking 1962; Jacobs 1968, 1969; Johri 1972; Lal 1977; Matzner 1969; Misner 1967; Roy and Singh 1976; Singh and Abdussattar 1973; Stewart and Ellis 1968; Vajk and Elgrowth 1970).

In this paper, we consider Heckmann-Schücking models which contain an ideal fluid obeying a γ-law equation of state. The underlying idea is to investigate the spatially homogeneous and anisotropic perfect fluid cosmological solutions of the Einstein field equations. In this way, we have derived perfect fluid models of the anisotropic universe corresponding to $\gamma = 1, \frac{4}{3}, 2$, and $\frac{5}{3}$ that is for dust-filled, radiation dominated, super dense and non-relativistic gas-filled stages of the evolution of universe. Our models include LRS fluid models (Stewart and Ellis 1968) and the non-degenerate Petrov type I model. We choose units such that $8\pi G = c^4 = 1$. We have discussed various physical and geometrical properties of models.

2. THE FIELD EQUATIONS

We consider the general Heckmann-Schücking metric

$$ds^2 = dt^2 - A^2dx^2 - B^2dy^2 - C^2dz^2 \qquad ...(2.1)$$

where A, B, C are functions of time only. We consider energy momentum tensor for a perfect fluid, that is

$$T_{ij} = (p + \rho)\,\lambda_i\lambda_j - pg_{ij} \qquad ...(2.2)$$

with

$$g_{ij}\lambda^i\lambda^j = 1 \qquad ...(2.3)$$

where λ^i is the 4-velocity vector of the fluid and velocity of light is taken to be unity.

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The Einstein field equations are

$$R_{ij} - \tfrac{1}{2} R g_{ij} + \Lambda g_{ij} = -T_{ij} \qquad \text{...(2.4)}$$

Choosing comoving coordinates and neglecting the cosmological constant Λ, the field equations. (2.4) in terms of line element (2.1) can be written down as

$$\left(\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC}\right) = -p \qquad \text{...(2.5)}$$

$$\left(\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC}\right) = -p \qquad \text{...(2.6)}$$

$$\left(\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB}\right) = -p \qquad \text{...(2.7)}$$

$$\left(\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{AC}\right) = \rho \qquad \text{...(2.8)}$$

where A_4, B_4, C_4 stand for time derivatives of A, B, C respectively. The mass-energy conservation equation $T^{ij}_{;j} = 0$ gives

$$\rho_4 = -(p + \rho)\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) \qquad \text{...(2.9)}$$

Taking the equation of state for barotropic fluids

$$p = (\gamma - 1)\,\rho \qquad \text{...(2.10)}$$

Equation (2.9) yields

$$\rho_4/\rho = -\gamma (ABC)_4/ABC \qquad \text{...(2.11)}$$

on integration, we get

$$\rho = L/(ABC)^{\gamma} \qquad \text{...(2.12)}$$

where L is an arbitrary constant of integration.

Now adding eqns. (2.5), (2.6), (2.7) to 3-times of (2.8), we get

$$\left(\frac{(ABC)_4}{ABC}\right)_4 + \left(\frac{(ABC)_4}{ABC}\right)^2 = \tfrac{3}{2}(\rho - p). \qquad \text{...(2.13)}$$

By virtue of eqns. (2.10), (2.11), (2.12), eqn. (2.13) takes the form

$$(\rho^{-1/\gamma})_{44} = -\tfrac{3}{2}(\gamma - 2)\,(\rho^{-1/\gamma})^{(1-\gamma)} \qquad \text{...(2.14)}$$

On substitution

$$X = \rho^{-1/\gamma} \qquad \text{...(2.15)}$$

eqn. (2.14) takes the form

$$X_{44} = -\tfrac{3}{2}(\gamma - 2)\, X^{(1-\gamma)} \qquad \text{...(2.16)}$$

which has the first integral

$$X_4^2 = 3X^{(2-\gamma)} + K \qquad \text{...(2.17)}$$

where K is an arbitrary constant of integration.

Further integration yields

$$\int \frac{dx}{(3X^{(2-\gamma)} + K)^{1/2}} = t + M \qquad \text{...(2.18)}$$

where M is an arbitrary constant of integration.

Expansion scalar and shear scalar can be obtained with the help of eqns. (2.8) and (2.4), as such:

$$\text{Expansion scalar } \theta = \tfrac{1}{2}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right)$$

$$\theta = -\tfrac{1}{2}\rho_4/\gamma\rho. \qquad \text{...(2.19)}$$

Shear scalar is given by

$$3\sigma^2 = (\theta_{11} + \theta_{22} + \theta_{33})^2 - 3(\theta_{11}\theta_{22} + \theta_{22}\theta_{33} + \theta_{33}\theta_{11})$$

or

$$\sigma^2 = \tfrac{1}{3}\left(\frac{\rho_4^2}{\gamma^2\rho^2} - 3\rho\right). \qquad \text{...(2.20)}$$

From eqn. (2.5) and (2.6), we get

$$\frac{(A/B)_4}{(A/B)} = \frac{K_1}{(ABC)}. \qquad \text{...(2.21)}$$

Similarly

$$\frac{(B/C)_4}{(B/C)} = \frac{K_2}{ABC} \qquad \text{...(2.22)}$$

and

$$\frac{(C/A)_4}{(C/A)} = \frac{K_3}{ABC} \qquad \text{...(2.23)}$$

where the constants of integration K_1, K_2 and K_3 satisfy the relation

$$K_1 + K_2 + K_3 = 0. \qquad \text{...(2.24)}$$

Now we consider the cases of the dust-filled, radiation dominated and superdense stages of the universe one by one.

3. Case I — Dust-filled Universe

This case corresponds to incoherent matter for which $p = 0$. Therefore putting $\gamma = 1$ in eqn. (2.18) and integrating we get

$$X = \tfrac{3}{4}(t + M)^2 - \frac{K}{3}$$

where K is an arbitrary constant of integration.

Inserting the value of X from eqn. (2.15), we get

$$\tfrac{4}{3}\rho^{-1} = (t + M)^2 - \tfrac{4}{9}K. \qquad \text{...(3.1)}$$

Assuming the occurrence of primordial singularity as in all relativistic models, we have $\rho \to \infty$ as $t \to 0$.

As such $\quad \tfrac{4}{9}K = M^2.$

Inserting the value of M in eqn. (3.1) we have

$$\rho = \frac{4}{3(t^2 + 2Mt)}. \qquad \text{...(3.2)}$$

The same result was obtained by Johri (1972) and Bandopadhyay (1977) by tetrad approach. By virtue of eqn. (2.12) we get

$$\frac{L}{ABC} = \frac{4}{3(t^2 + 2Mt)}. \qquad \text{...(3.3)}$$

On integrating eqns. (2.21), (2.22) and (2.23) respectively we have

$$A/B = P\left\{\frac{t}{t + 2M}\right\}^{p_1} \qquad \text{...(3.4)}$$

$$B/C = Q\left\{\frac{t}{t + 2M}\right\}^{p_2} \qquad \text{...(3.5)}$$

and

$$C/A = N\left\{\frac{t}{t + 2M}\right\}^{p_3} \qquad \text{...(3.6)}$$

where P, Q, N, p_1, p_2 and p_3 are arbitrary constants satisfying $PQN = 1$ and

$$p_1 + p_2 + p_3 = 0.$$

Combining eqns. (3.3), (3.4) and (3.5), we get

$$\left.\begin{aligned} A^3 &= A_0^3\left(\frac{t}{t + 2M}\right)^{(p_1 - p_3)}(t^2 + 2Mt) \\ B^3 &= B_0^3\left(\frac{t}{t + 2M}\right)^{-(2p_1 + p_3)}(t^2 + 2Mt) \\ C^3 &= C_0^3\left(\frac{t}{t + 2M}\right)^{(p_1 + 2p_3)}(t^2 + 2Mt) \end{aligned}\right\} \qquad \text{...(3.7)}$$

where A_0, B_0 and C_0 are arbitrary constants. Thus we get the following metric for dust-filled spatially homogeneous and anisotropic universe

$$ds^2 = dt^2 - (t^2 + 2Mt)^{2/3} \left\{ A_0^2 \left(\frac{t}{t+2M}\right)^{2(p_1-p_3)/3} dX^2 + B_0^2 \left(\frac{t}{t+2M}\right)^{-2(2p_1+p_3)/3} dY^2 + C_0^2 \left(\frac{t}{t+2M}\right)^{2(p_1+2p_3)/3} dZ^2 \right\}.$$

Equation (3.7) can be rewritten as

$$A = \left(\frac{t^{2/3}}{S}\right)^{(p_1-p_3)} . S$$

$$B = \left(\frac{t^{2/3}}{S}\right)^{-(2p_1+p_3)} . S$$

$$C = \left(\frac{t^{2/3}}{S}\right)^{(p_1+2p_3)} . S$$

where $S^3 = (t^2 + 2Mt)$ and $A_0 = B_0 = C_0 = 1$.

This corresponds to the solution obtained by Hawking and Ellis (1973). This model is characterised by the following properties:

(i) The density ρ is given by eqn.(3.2). It is obvious that the density of matter decreases more rapidly in Heckmann-Schücking case as compared to density $\rho = K/t^2$ in Friedmann model.

(ii) This model corresponds to LRS model when we put $p_1 = -p_3$ and corresponds to non-degenerate Petrov type-I universe when we put $p_1 = p_3$.

(iii) The expansion scalar is given by

$$\theta = -\tfrac{1}{3}\rho_4/\rho$$

or

$$= \tfrac{2}{3}(t + M)/(t^2 + 2Mt).$$

This shows that expansion goes on decreasing more rapidly with time.

(iv) The shear scalar is given by

$$\sigma^2 = 4M^2/3(t^2 + 2Mt)^2.$$

(v) The relative anisotropy in this model is given by

$$\frac{\sigma^2}{\rho} = M^2/(t^2 + 2Mt).$$

This parameter is the ratio of the anisotropic energy density of matter (except for the collisionless radiation) to the total energy density of the universe.

The above equation shows that in dust-filled universe, anisotropic energy density decreases more rapidly with time in comparison with total energy density.

4. Case II — Radiation Dominated Universe

In this case $\gamma = 4/3$, i.e. $p = \rho/3$.

Equation (2.17) takes the form

$$\rho_4 = \frac{4}{\sqrt{3}} \rho^{3/2} U$$

where

$$U = (1 + \tfrac{1}{3} K\rho^{1/2})^{1/2}. \qquad \ldots(4.1)$$

This equation yields on integration (Dwight 1957c)

$$\frac{K}{2\sqrt{3}}\left[\frac{3}{K} \rho^{-1/2}U + \tfrac{1}{2} \log\left(\frac{U-1}{U+1}\right)\right] = t + M \qquad \ldots(4.2)$$

where M is an arbitrary constant of integration and $K > 0$. Assuming the occurrence of primordial singularity as in all relativistic model, we have $\rho \to \infty$ as $t \to 0$.

Therefore $M = 0$.

Now conservation eqn. (2.12) yields

$$\rho = L/(ABC)^{4/3}.$$

Putting

$$S^3 = ABC \qquad \ldots(4.3)$$

we get

$$\rho = L/S^4. \qquad \ldots(4.4)$$

Inserting the value of ρ and M, eqn. (4.2) becomes

$$\frac{K}{2\sqrt{3}}\left[\frac{3}{K} L^{-1/2}S^2V + \tfrac{1}{2} \log\left(\frac{V-1}{V+1}\right)\right] = t \qquad \ldots(4.5)$$

where

$$V = \left(1 + \frac{K}{3}\frac{L^{1/2}}{S^2}\right)^{1/2}. \qquad \ldots(4.6)$$

Since it is very intricate to express S in terms of t, we express A, B, C in terms of S. From eqns. (4.1) and (4.4), we have

$$\rho_4 = 4LS_4/S^5 \qquad \ldots(4.7)$$

and

$$S_4 = \frac{LV}{\sqrt{3}\,S}. \qquad \ldots(4.8)$$

We can rewrite eqn. (2.21) in the form

$$\frac{d}{dS}(\log(A/B)) = K_1/S^3S_4. \qquad \ldots(4.9)$$

Inserting the value of S_4 from eqn. (4.8) into eqn. (4.9) and integrating (Dwight 1957a), we get

$$A/B = r_1 W^{p_1} \qquad \ldots(4.10)$$

where r_1, p_1 are arbitrary constants and

$$W = \left(\frac{SV + \frac{1}{3}K^{1/2}L^{1/4}}{SV - \frac{1}{3}K^{1/2}L^{1/4}}\right). \qquad \ldots(4.11)$$

Similarly

$$B/C = r_2 W^{p_2} \qquad \ldots(4.12)$$

and

$$C/A = r_3 W^{p_3} \qquad \ldots(4.13)$$

where r_2, r_3, p_2, p_3 are arbitrary constants satisfying $r_1 r_2 r_3 = 1$ and

$$p_1 + p_2 + p_3 = 0.$$

Equations (4.3), (4.10), (4.12) and (4.13), ultimately give

$$A = A_0 W^{(p_1 - p_3)/3} . S, \quad B = B_0 W^{-(2p_1 + p_3)/3} . S, \quad C = C_0 W^{(p_1 + 2p_3)/3} . S \qquad \ldots(4.14)$$

where A_0, B_0 and C_0 are arbitrary constants satisfying $A_0 . B_0 . C_0 = 1$.

This model is characterized by the following properties

(i) The density ρ is given by eqn. (4.5).

(ii) The expansion scalar is given by

$$\theta = \tfrac{1}{4}\rho_4/\rho$$
$$= \left(\frac{L}{B_3}\right)^{1/2}\frac{V}{S^2}.$$

(iii) The shear scalar is given by

$$\sigma^2 = \tfrac{1}{3}\left\{\frac{9}{16}\left(\frac{\rho_4}{\rho}\right)^2 - 3\rho\right\} = \frac{N}{S^6}$$

where N is an arbitrary constant.

(iv) Like previous model this model also corresponds to LRS model and non-degenerate Petrov type-I universe, when we put $p_1 = -p_3$ and $p_1 = p_3$ respectively.

(v) The relative anisotropy

$$\frac{\sigma^2}{\rho} = \frac{P}{S^2}$$

where P is an arbitrary constant.

This equation shows that in radiation dominated universe, anisotropic energy density decreases more rapidly with time in comparison with the total energy density of the universe.

5. Case III — Super Dense Universe

Zeldovich gave the equation of state for stiff matter by choosing $\gamma = 2$; this implies $p = \rho$. Therefore, inserting the value of γ in eqn. (2.17) and integrating we get $X = (\alpha t + \beta)$

where α, β are arbitrary constants or $1 = \rho(\alpha t + \beta)^{-2}$.

Now singularity condition $\rho \to \infty$ as $t \to 0$ implies $\beta = 0$.

Therefore we have

$$\rho = k/t^2 \qquad \text{...(5.1)}$$

where k is an arbitrary constant.

From eqns. (2.12) and (5.1) we get

$$ABC = mt \qquad \text{...(5.2)}$$

where m is an arbitrary constant.

Now proceeding exactly as in the dust case, we have

$$A/B = r_1 t^{k_1}, \quad B/C = r_2 t^{k_2}, \quad C/A = r_3 t^{k_3} \qquad \text{...(5.3)}$$

where r_1, r_2, r_3, k_1, k_2 and k_3 are arbitrary constants satisfying $r_1 . r_2 . r_3 = 1$ and

$$k_1 + k_2 + k_3 = 0.$$

These equations ultimately give

$$A = A_0 t^{p_1}, \quad B = B_0 t^{p_2}, \quad C = C_0 t^{p_3} \qquad \text{...(5.4)}$$

where A_0, B_0, C_0, p_1, p_2 and p_3 are arbitrary constants satisfying $A_0 . B_0 . C_0 = m$ and

$$p_1 + p_2 + p_3 = 1 \qquad \text{...(5.5)}$$

Moreover from eqns. (2.8), (5.4) and (5.5) we have

$$\Sigma\, p_i^2 = p_1^2 + p_2^2 + p_3^2 = 1 - 2K \qquad \text{...(5.6)}$$

which implies $\Sigma\, p_i^2 \leqslant 1$ (since K is positive).

The components of the expansion tensor are

$$\theta_{11} = p_1/t, \;\; \theta_{22} = p_2/t, \;\; \theta_{33} = p_3/t.$$

Therefore expansion scalar

$$\theta = \tfrac{1}{3}(\theta_{11} + \theta_{22} + \theta_{33}) = \frac{1}{3t}. \qquad \ldots(5.7)$$

And shear scalar is given by

$$\sigma^2 = \tfrac{1}{3}\,[(\theta_{11} + \theta_{22} + \theta_{33})^2 - 3(\theta_{11}\theta_{22} + \theta_{22}\theta_{33} + \theta_{33}\theta_{11})]$$

$$= \frac{1 - 3(p_1 p_2 + p_2 p_3 + p_3 p_1)}{3t^2} \qquad \ldots(5.8)$$

Now σ will be real provided

$$(p_1 p_2 + p_2 p_3 + p_3 p_1) \leqslant \tfrac{1}{3}$$

or $\quad \Sigma\, p_i^2 \geqslant \tfrac{1}{3}$

Thus we get the following metric for super dense stage of the universe

$$ds^2 = dt^2 - A_0^2\, t^{2p_1}\, dx^2 - B_0^2\, t^{2p_2}\, dy^2 - C_0 t^{2p_3}\, dz^2 \qquad \ldots(5.9)$$

where p_1, p_2 and p_3 satisfy

$$\Sigma\, p_i = 1$$

and

$$1 \geqslant \Sigma\, p_i^2 \geqslant \tfrac{1}{3}. \qquad \ldots(5.10)$$

The model (5.9) is obviously the generalization of well known Kasner model of empty space in which the constants p_i's satisfy $\Sigma\, p_i = 1$, $\Sigma\, p_i^2 = 1$ and $A_0 = B_0 = C_0 = 1$. The limits of $\Sigma\, p_i^2$ obtained by us also agree with restrictions prescbribed by Lal (1977) for non-empty universe.

The metric (5.9) is also transformable to the metric obtained by Roy and Singh (1976)

$$ds^2 = L^2 \tau^{1-a^2}(d\tau - dX^2) - \tau^{1+a} dY^2 - \tau^{1-a^2}\, dZ^2$$

by using the following coordinate transformation

$$A_0 d\tau \rightarrow t^{-p_1}\, dt, \quad B_0\,\{A_0(1 - p_1)\}^{p_2/(1-p_1)}\, dy \rightarrow dY,$$

$$C_0\,\{A_0(1 - p_1)\}^{p_3/(1-p_1)}\, dz \rightarrow dZ, \quad L = A_0\,\{A_0(1 - p_1)\}^{p_1/(1-p_1)},$$

$$a^2 = (3p_1 - 1)/(p_1 - 1), \quad p_2 = \tfrac{1}{2}(1 - p_1)(1 + a)$$

and $$p_3 = \tfrac{1}{2}(1 - p_1)(1 - a)$$

The relative anisotropy in the model is given by

$$\sigma^2/\rho = K'$$

where K' is an arbitrary constant.

This shows that unlike the cases of the dust-filled and radiation dominated universe, the anisotropic energy density in this case is proportional to total energy density of the universe.

6. Case IV — Non-relativistic Gas-filled Universe

In this case $\gamma = \frac{5}{3}$ i.e. $p = \frac{2}{3}\rho$

Equation (2.17) takes the form

$$\frac{dX}{(3X + K)^{1/2}} = t + M \qquad \text{...(6.1)}$$

This equation yields on integration (Dwight 1957b)

$$\tfrac{2}{9}(3\rho^{-1/5} + K)^{1/2}\left[(3\rho^{-1/5} + K)^2 - \frac{2K}{3}(3\rho^{-1/5} + K) + K^2\right] = t + M \qquad \text{...(6.2)}$$

where M is an arbitrary constant of integration and $K > 0$.

As $t \to 0$, $\rho \to \infty$ therefore $M = 0$.

Now conservation eqn. (2.12) yields

$$\rho = L/(ABC)^{5/3}. \qquad \text{...(6.3)}$$

Putting $S^3 = (ABC)$ we get

$$\rho = L/S^5. \qquad \text{...(6.4)}$$

Inserting the value of ρ and M, eqn. (6.2) becomes

$$\frac{2}{9} V^{1/2}\left[V^2 - \frac{2K}{3} V + K^2\right] = t \qquad \text{...(6.5)}$$

where

$$V = (3L^{-1/5}S + K). \qquad \text{...(6.6)}$$

Proceeding exactly in same manner as in the case of radiation filled universe we get

$$A = A_0 U^{p_1} S, \quad B = B_0 U^{p_2} S, \quad C = C_0 U^{p_3} S \qquad \ldots(6.7)$$

where A_0, B_0, C_0, p_1, p_2, p_3 are arbitrary constants satisfying $p_1 + p_2 + p_3 = 0$ and $A_0 B_0 C_0 = 1$ and

$$U = (p - q)/(p + q), \quad p = (3L^{-1/5}S + K)^{1/2}$$

$$q = K^{1/2}.$$

This model is characterized by the following properties:

(i) The density ρ is given by eqn. (6.5).

(ii) The expansion scalar is given by

$$\theta = -\tfrac{1}{5}\rho_4/\rho$$

$$= \frac{L^{3/5}}{S^3}(3L^{-1/5}S + K)^{1/2}.$$

(iii) The shear scalar is given by

$$\sigma^2 = \tfrac{1}{3}\{9/25(\rho_4/\rho)^2 - 3\rho\}$$

$$= \frac{L}{3S^5}\left\{(3L^{-1/5}S + K)\frac{L^{1/5}}{S} - 3\right\}.$$

(iv) The relative anisotropy

$$\frac{\sigma^2}{\rho} = \tfrac{1}{3}\left\{(3L^{-1/5}S + K)\frac{L^{1/5}}{S} - 3\right\}.$$

This equation shows that in non-relativistic gas-filled universe anisotropic energy density decreases rapidly with time in comparison with total energy density of universe.

(v) Like previous model this model also corresponds to LRS model and non-degenerate Petrov type-I universe, when we put $p_1 = -p_3$ and $p_1 = p_3$ respectively.

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