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ON THE MEASURABILITY AND CONTINUITY PROPERTIES OF THE COSINE OPERATOR

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Let X be a Banach space and $B(X)$ denote the family of bounded linear operators on X. Let $R^+ = [0, \infty)$ and I denote the identity operator. A one parameter family $\{C(t); t \in R^+\}$, $C : R^+ \rightarrow B(X)$, is said to be a cosine operator family on X, if (i) $C(0) = I$, and (ii) $C(t + s) + C(t - s) = 2C(t)\, C(s)$, $t, s \in R^+, s \leqslant t$. In this paper, it is proved that the Lebesgue measurability of $\{C(t) f\}$ on $R^+ - \{0\}$ implies continuity of $\{C(t) f\}$ on $R^+ - \{0\}$, for every $f \in X$.

1. INTRODUCTION

Let X be a Banach space and $B(X)$ denote the family of bounded linear operators on X. Let $R^+ = [0, \infty)$ and let I denote the identity operator. A one parameter family of operators $\{C(t); t \in R^+\}$, $C : R^+ \rightarrow B(X)$, is said to be a cosine operator family on X (refer Sova 1966), if

$$\left.\begin{array}{ll} \text{(i)} & C(0) = I, \quad \text{and} \\ \text{(ii)} & C(t + s) + C(t - s) = 2C(t)\, C(s),\ s \leqslant t,\ s, t \in R^+. \end{array}\right\} \quad ...(1)$$

Kurepa (1960) has discussed the connection between the weak measurability and weak continuity of the cosine operator family on separable and reflexive Banach space. In this paper we shall establish that the strong measurability of the cosine operator family on a Banach space implies its strong continuity.

2. THE MEASURABILITY, BOUNDEDNESS AND CONTINUITY PROPERTIES PROPOSITION

Let $\{C(t); t \in R^+\}$, $C : R^+ \rightarrow B(X)$, satisfy the eqn. (1). Then the Lebesgue measurability of $C(.)f$ on $R^+ - \{0\}$ implies continuity of $C(.)f$ on $R^+ - \{0\}$.

PROOF : We shall be first proving that the Lebesgue measurability of $C(.)f$ on $R^+ - \{0\}$ implies the boundedness of $C(.)f$ on every closed interval $[a, b]$ of $R^+ - \{0\}$, for a fixed $f \in X$. Then the boundedness of $C(.)f$ is used to prove its continuity on $R^+ - \{0\}$.

The Lebesgue measurability of $C(.)f$ on $R^+ - \{0\}$ implies the Lebesgue measurability of $\| C(.)f \|$ on $R^+ - \{0\}$, (cf. Hille and Phillips 1957). Suppose, to the contrary that $\| C(.)f \|$ is not bounded on a closed interval $[a, b] \subset R^+ - \{0\}$. Then there exists a $\tau \in [a, b]$ and a sequence $\tau_n \in [a, b]$, such that $\tau_n \uparrow \tau$ and

$$\| C(\tau_n) f \| \geqslant n, \; n = 1, 2, 3, \ldots . \qquad \ldots(2)$$

On the other hand $\| C(.)f \|$ being measurable, there exists a constant $M > 0$ and a Lebesgue measurable set $G \subset [0, \tau]$ with

$$m(G) > \tfrac{3}{4}\tau \qquad \ldots(3)$$

such that

$$\sup_{t \in G} \| C(t)f \| \leqslant M. \qquad \ldots(4)$$

Let

$$A_k = \tfrac{1}{2}\tau_k - \tfrac{1}{2}(G \cap [0, \tau_k])$$

and

$$B_k = G \cap [0, \tfrac{1}{2}\tau_k].$$

Define

$$A = \tfrac{1}{2}\tau - \tfrac{1}{2}(G \cap [0, \tau])$$

and

$$B = G \cap [0, \tfrac{1}{2}\tau].$$

We claim that $m(A \cap B) > 0$. If possible, suppose that $m(A \cap B) = 0$, then $m(A) + m(B) \leqslant \frac{1}{2}\tau$, because $A \subset [0, \frac{1}{2}\tau]$ and $B \subset [0, \frac{1}{2}\tau]$. But $m(A) = \frac{1}{2}m(G)$. Since $m(A) + m(B) \leqslant \frac{1}{2}\tau$, we have $m(G) + 2m(B) \leqslant \tau$. Hence $\frac{3}{4}\tau < m(G) \leqslant \tau - 2m(B)$, that is

$$m(B) < \tfrac{1}{8}\tau. \qquad \ldots(5)$$

Also

$$G = (G \cap [0, \tfrac{1}{2}\tau]) \cup (G \cap [\tfrac{1}{2}\tau, \tau])$$

$$= B \cup C, \quad \text{where} \quad C = G \cap [\tfrac{1}{2}\tau, \tau].$$

Therefore

$$m(G) = m(B) + m(C), \quad \text{and} \quad m(C) \leqslant \tfrac{1}{2}\tau.$$

So

$$\tfrac{3}{4}\tau < m(G) = m(B) + m(C) \leqslant m(B) + \tfrac{1}{2}\tau$$

that is,

$$m(B) > \tfrac{1}{4}\tau. \qquad \ldots(6)$$

But (5) and (6) contradict each other, hence our claim is proved, that is, $m(A \cap B) > 0$. Then there exists a real $\delta > 0$ such that $m(A \cap B) > \delta$.

Let

$$E = A \cap B, \quad E_n = A_n \cap B_n$$

and

$$H_n = \{\tau_n - \eta;\, \eta \in E_n\}, \quad n = 1, 2, 3, \dots . \qquad \dots(7)$$

Now $E_n \to E$; therefore, for sufficiently large n, $m(E_n) > 0$ and for such values of n, if $\eta \in E_n$, one can easily see that η and $\tau_n - 2\eta$ both belong to G. Similar result hold for E and G in place of E_n and G.

Clearly H_n is measurable, and for sufficiently large n, $m(H_n) \geqslant \frac{1}{2}\delta$. Now for $\eta \in E_n$, we have, using eqn. (1),

$$\begin{aligned} n &\leqslant \| C(\tau_n) f \| = \| C((\tau_n - \eta) + \eta) f \| \\ &= \| 2C(\tau_n - \eta)\, C(\eta) f - C(\tau_n - 2\eta) f \| \\ &\leqslant 2 \| C(\tau_n - \eta) \| \cdot \| C(\eta) f \| + \| C(\tau_n - 2\eta) f \| \\ &\leqslant 2M \| C(\tau_n - \eta) \| + M. \end{aligned}$$

Hence

$$\frac{n - M}{2M} \leqslant \| C(\tau_n - \eta) \|, \quad \text{for} \quad \eta \in E_n$$

that is,

$$\frac{n - M}{2M} \leqslant \| C(t) \|, \quad \text{for} \quad t \in H_n.$$

Denoting $\limsup H_n$ by H, we see that $m(H) \geqslant \frac{1}{2}\delta$ and $\| C(t) \| = \infty$ for all $t \in H$, which contradicts that $\{C(t);\, t \in R^+\}$ is a family of bounded linear operators.

This establishes the boundedness of $\| C(t) f \|$ on every closed interval of $R^+ - \{0\}$, for every $f \in X$. The proof that the boundedness of $\| C(t) f \|$ on every closed interval implies continuity of $\{C(t) f\}$ for $t > 0$ follows on the same lines as that of semigroup of operators (refer Hille and Phillips 1957).

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