Indian J. pure appl. Math., 12(1) : 84–87, January 1981

A NOTE ON A DISCONTINUOUS HYPERGEOMETRIC INTEGRAL

HAROLD EXTON

27 Hollinhurst Avenue, Penwortham, Preston, Lancashire PR1 OAE, U.K.

(*Received* 23 *September* 1980)

The Mellin transform of the product of a pair of confluent hypergeometric functions is evaluated and it is shown that the result is a discontinuous function of the arguments of these hypergeometric functions.

A well-known discontinuous integral which arises in the theory of Bessel functions is the Weber-Schafheitlin integral

$$\int_0^\infty t^{\alpha+\beta-\gamma} J_{\alpha-\beta}(at)\, J_{\gamma-1}(bt)\, dt \qquad \ldots(1)$$

in which a and b are positive so as to ensure the convergence at the upper limit (see Watson 1944, p. 398). If $0 < b < a$, the integral (1) takes the form

$$\frac{b^{\gamma-1}\Gamma(\alpha)}{2^{\gamma-\alpha-\beta}a^{\alpha+\beta}\Gamma(\gamma)\,\Gamma(1-\beta)}\, {}_2F_1(\alpha, \beta; \gamma; b^2/a^2) \qquad \ldots(2)$$

whereas, on the other hand, if the quantities a and b and also $\alpha - \beta$ and $\gamma - 1$ are interchanged, we have

$$\frac{a^{\alpha-\beta}(\alpha)}{2^{\gamma-\alpha-\beta}b^{2\alpha-\gamma+1}\Gamma(\gamma-\alpha)\,\Gamma(\alpha-\beta+1)}\, {}_2F_1(\alpha, \alpha-\gamma+1; \alpha-\beta+1; a^2/b^2). \qquad \ldots(3)$$

The expressions (2) and (3) are not the analytic continuations of the same function, so that a discontinuity of the integral (1) exists when $a = b$. In fact, when $a = b$, we have the result

$$\int_0^\infty t^{\alpha+\beta-\gamma} J_{\alpha-\beta}(at)\, J_{\gamma-1}(at)\, dt = \frac{(\tfrac{1}{2}a)^{\gamma-\alpha-\beta-1}\Gamma(\gamma-\alpha-\beta)\,\Gamma(\alpha)}{2\Gamma(1-\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad \ldots(4)$$

provided that Re (α) and Re $(\gamma - \alpha - \beta) > 0$.

In this paper, an integral similar to (1),

$$\int_0^\infty t^{a-1}\, {}_1F_1(b; c; -xt)\, {}_1F_1(b'; c'; -yt)\, dt \qquad \ldots(5)$$

and which is also discontinuous, is investigated. First of all, consider the well-known formula

$$\int_0^\infty e^{-pt} t^{a-1} {}_1F_1(b; c; -xt)\, {}_1F_1(b'; c'; -yt)\, dt$$

$$= p^{-1}\Gamma(a)\, F_2(a, b, b'; c, c'; -x/p, -y/p) \qquad \ldots(6)$$

where it is taken that $\mathrm{Re}\,(p) > 0$ initially (see Olsson 1977, p. 1283). The Appell function F_2 is defined by the series representation

$$F_2(a, b, b'; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)\,(b, m)\,(b', n)\, x^m y^n}{(c, m)\,(c', n)\, m!\, n!}$$

which converges for $|x| + |y| < 1$ (see Appell and Kampé de Fériet 1926, p. 14). The Pochhammer symbol (a, m) is defined by the expression

$$(a, m) = a(a+1)(a+2)\ldots(a+m-1) = \Gamma(a+m)/\Gamma(a),\ (a, 0) = 1.$$

The integral under consideration, (5), may be obtained from (6) by taking the limit as $p \to 0$. In order to facilitate this calculation, we obtain an analytic continuation formula for the function F_2. Expand this function as a series of Gauss hypergeometric functions :

$$F_2(a, b, b'; c, c'; x, y) = \sum_{m=0}^{\infty} \frac{(a, m)\,(b, m)\, x^m}{(c, m)\, m}\, {}_2F_1(a+m, b'; c'; y) \qquad \ldots(7)$$

and apply the formula

$$\begin{aligned} {}_2F_1(a+m, b'; c'; y) &= \frac{\Gamma(c')\,\Gamma(b'-a-m)}{\Gamma(b')\,\Gamma(c'-a-m)}\,(-y)^{-a-m} \\ &\quad \times {}_2F_1(a+m, 1-c'+a+m; 1-b'+a+m; 1/y) \\ &\quad + \frac{\Gamma(c')\,\Gamma(a-b'+m)}{\Gamma(a+m)\,\Gamma(c'-b')}(-y)^{-b'}\, {}_2F_1(b', 1-c'+b'; 1-a-b'-m; 1/y) \end{aligned} \qquad \ldots(8)$$

(Whittaker and Watson 1952, p. 289).

The series on the right-hand side of (7) splits into two parts. After simplification, the first part becomes

$$\frac{\Gamma(c')\,\Gamma(b'-a)}{\Gamma(b')\,\Gamma(c'-a)}(-y)^{-a} \sum_{m,n=0}^{\infty} \frac{(a, m+n)\,(1-c'+a, m+n)\,(b, m)}{(1-b'+a, m+n)\,(c, m)\, m!\, n!}$$

$$\times (-x/y)^m\, (1/y)^n \qquad \ldots(9)$$

and this is in a suitable form. The second part requires two further applications of (8) in order that the resulting expressions will assume convenient forms. After some reduction, it is found that

$$
\begin{aligned}
&F_2(a, b, b'; c, c'; x, y) \\
&= \frac{\Gamma(c')\,\Gamma(b'-a)}{\Gamma(b')\,\Gamma(c'-a)}(-y)^{-a}\sum_{m,n=0}^{\infty}\frac{(a, m+n)(1-c'+a, m+n)(b, m)}{(1-b'+a, m+n)(c, m)\,m!\,n!} \\
&\quad\times(-x/y)^m\,(1/y)^n \\
&\quad+\frac{\Gamma(c')\,\Gamma(c)\,\Gamma(a-b')\,\Gamma(b+b'-a)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c'-b')\,\Gamma(c+b'-a)}(-y)^{-b'}(-x)^{b'-a} \\
&\quad\times\sum_{m,n=0}^{\infty}\frac{(a-b', m-n)(1-b'-c+a, m-n)(b', n)(1-c'+b', n)}{(1-b-b'+a, m-n)\,m!\,n!} \\
&\quad\times(1/x)^m\,(-x/y)^n+\frac{\Gamma(c')\,\Gamma(c)\,\Gamma(a-b-b')}{\Gamma(a)\,\Gamma(c'-b')\,\Gamma(c-b)}(-y)^{-b}(-x)^{-b'} \\
&\quad\times\sum_{m,n=0}^{\infty}\frac{(b, m)(1-c+b, m)(b', n)(1-c'+b', n)}{(1+b+b'-a, m+n)\,m!\,n!}(1/x)^m\,(1/y)^n.
\end{aligned}
\qquad \ldots(10)
$$

The series on the right-hand side of (10) may be recognised as being of the types F_P, F_Q and F_3, respectively (see Olsson 1977, pp. 1286 & 1287).

The required limit may now be taken, and we have

$$
\begin{aligned}
&\int_0^{\infty} t^{a-1}\,{}_1F_1(b, c, -xt)\,{}_1F_1(b'; c'; -yt)\,dt \\
&\quad=\frac{\Gamma(a)\,\Gamma(c')\,\Gamma(b'-a)}{\Gamma(b')\,\Gamma(c'-a)}y^{-a}\,{}_3F_2(a, 1-c'+a, b; 1-b'+a, c; -x/y) \\
&\qquad+\frac{\Gamma(c')\,\Gamma(c)\,\Gamma(a-b')\,\Gamma(b+b'-a)}{\Gamma(b)\,\Gamma(c'-b')\,\Gamma(c+b'-a)}y^{-b'}x^{b'-a} \\
&\qquad\times{}_3F_2(b+b'-a, 1-c'+b', b'; 1+b'-a, c+b'-a; x/y)
\end{aligned}
\qquad \ldots(11)
$$

if $0 < x < y$. By symmetry, if $0 < y < x$,

$$
\int_0^{\infty} t^{a-1}\,{}_1F_1(b; c; -xt)\,{}_1F_1(b';c'; -yt)\,dt =
$$

(equation continued on p. 87)

$$= \frac{\Gamma(a)\,\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,x^{-a}\,{}_3F_2(a,\,1-c+a,\,b';\,1-b+a,\,c';\,-y/x)$$

$$+\frac{\Gamma(c')\,\Gamma(c)\,\Gamma(a-b)\,\Gamma(b+b'-a)}{\Gamma(b')\,\Gamma(c-b)\,\Gamma(c'+b-a)}\,x^{-b}y^{b-a}$$

$$\times\,{}_3F_2(b+b'-a,\,1-c+b,\,b;\,1+b-a,\,c'+b-a;\,y/x). \qquad \ldots(12)$$

The discussion of the critical case of the integral (5) when $x = y$ is more involved, and this problem is reserved for future study.

References

Appell, P., and Kampé de Fériet, J. (1926). Fonctions Hypérgéometriques et Hypérsphériques. Gautier Villars, Paris.

Olsson, P. O. M. (1977). On the integration of the differential equations of the five-parametric double-hypergeometric functions of the second order. *J. Math. Phys.*, **18**, 1285–94.

Watson, G. N. (1944). The Theory of Bessel Functions, 2nd ed. Cambridge University Press, London.

Whittaker, E. T., and Watson, G. N. (1952). Modern Analysis, 4th ed. Cambridge University Press, London.