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$[J, f(x)]$ SUMMABILITY OF LEGENDRE SERIES

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In this paper, the domain of summability of the series of Legendre polynomials by the $[J, f(x)]$ method due to Jakimovski (1960) is obtained.

1. INTRODUCTION

The summability of series of Legendre polynomials has been investigated by Cowling and King (1962-63), Jakimovski (1964, 1966), King (1968), Powell (1966, 1967), Prachar (1948-49), Swaminathan (1976) and others. In section 2 of this note, we study the summability of such series by the $[J, f(x)]$ method due to Jakimovski (1960). The corresponding results for the Borel exponential method (cf. King 1968) and the generalized Abel method follow as particular cases from the main result of this note.

After Jakimovski (1960), we may define the $[J, f(x)]$ method of summability as follows : Let $f(x)$ be differentiable infinitely often in $[0, \infty)$. With a given sequence $\{s_n\}$ $(n = 0, 1, \ldots)$, associate the transform :

$$t(x) \equiv \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} f^{(n)}(x)\, s_n \equiv \sum_{n=0}^{\infty} a_n(x)\, s_n,\ x \geqslant 0. \qquad \ldots(1.1)$$

If $t(x)$ converges for all large x, and if $\lim t(x) = s$, $x \to \infty$, then s is called the $[J, f(x)]$-limit of the sequence $\{s_n\}$. Jakimovski (1960) proved that the $[J, f(x)]$ transform is regular if and only if

$$f(x) = \int_0^{\infty} e^{-ux}\, d\alpha(u),\ \ x \geqslant 0 \qquad \ldots(1.2)$$

where $\alpha(u)$ is a function of bounded variation in $[0, \infty)$ and

$$\alpha(0) = \alpha(0+) = 0,\ \alpha(\infty-) = 1. \qquad \ldots(1.3)$$

The Borel exponential mean is obtained by putting $\alpha(u) = 0,\ 0 \leqslant u < 1$; $\alpha(u) = 1,\ 1 \leqslant u < \infty$. We have the classical Abel method when $\alpha(u) = 1 - e^{-u}$. The generalized Abel A_γ transform has an absolutely continuous weight function

$$\alpha(u) = [\Gamma(\gamma + 1)]^{-1} \int_0^{u} \nu^{\gamma} e^{-\nu}\, d\nu,\ \ \gamma > -1.$$

2. Summability of Series of Legendre Polynomials by the [J, f(x)] Method

Let $P_n(z)$ and $Q_n(w)$ denote, respectively, the Legendre polynomials of the first and second kind of the nth degree. The Laplace integral representations of $P_n(z)$ and $Q_n(w)$ are given by (Whittaker and Watson 1952, Ch. 15, pp. 313–19)

$$P_n(z) = \frac{1}{\pi} \int_0^\pi [\zeta(\phi)]^n \, d\phi \qquad \ldots(2.1)$$

and

$$Q_n(w) = \int_0^\infty [\tau(\nu)]^{-n-1} \, d\nu \qquad \ldots(2.2)$$

where

$$\zeta = \zeta(\phi) = z + (z^2 - 1)^{1/2} \cos \phi$$

and

$$\tau = \tau(\nu) = w + (w^2 - 1)^{1/2} \cosh \nu.$$

The branch of $(z^2 - 1)^{1/2}$ is so chosen that $z + (z^2 - 1)^{1/2}$ lies in the exterior of the unit circle. Write

$$s_n = s_n(z, w) = \sum_{k=0}^{n} (2k + 1) \, P_k(z) \, Q_k(w) \qquad \ldots(2.3)$$

and

$$d_n = d_n(z, w) = P_{n+1}(z) \, Q_n(w) - P_n(z) \, Q_{n+1}(w). \qquad \ldots(2.4)$$

We have, by Christoffel formula (Whittaker and Watson 1952, Chap. 15, p. 321),

$$\frac{1}{w - z} = s_n + (n + 1) \frac{1}{w - z} d_n. \qquad \ldots(2.5)$$

By Heine's theorem (Whittaker and Watson 1952, Chap. 15, p. 322), the sequence $\{s_n(z, w)\}$ converges to $(w - z)^{-1}$ in the interior of the ellipse with foci ± 1, and passing through w. The following theorem asserts that, under suitable conditions, $\{s_n(z, w)\}$ is summable by regular $[J, f(x)]$ method to $(w - z)^{-1}$ in a larger region.

Theorem 2.1 — Let $0 < \gamma < 1$, $0 < \beta$, and let

$$S(w, \gamma) = \left\{ z : \operatorname{Re}\left(\frac{\zeta(\phi)}{\tau(\nu)}\right) < \gamma, \; 0 \leqslant \phi \leqslant \pi, \; 0 \leqslant \nu \right\}$$

$$T(w, \beta) = \left\{ z : \left|\frac{\zeta(\phi)}{\tau(\nu)}\right| < \beta, \; 0 \leqslant \phi \leqslant \pi, \; 0 \leqslant \nu \right\}$$

and

$$R(w, \gamma, \beta) = S(w, \gamma) \cap T(w, \beta).$$

Then $\{s_n(z, w)\}$ is $[J, f(x)]$-summable to $(w - z)^{-1}$ provided that

$$z \in R(w, \gamma, \beta).$$

PROOF: Let

$$J(x) = \sum_{n=0}^{\infty} a_n(x)\, s_n$$

where $a_n(x)$ and s_n are respectively given by (1.1) and (2.3). Then, using (2.5), we obtain

$$J(x) = \sum_{n=0}^{\infty} a_n(x) \left[\frac{1}{w - z} - (n + 1)\frac{1}{w - z} d_n\right]$$

$$= \frac{1}{w - z} \sum_{n=0}^{\infty} a_n(x) - \frac{1}{w - z} \sum_{n=0}^{\infty} a_n(x)\,(n + 1)\, d_n.$$

An easy calculation now shows that

$$\sum_{n=0}^{\infty} a_n(x) = 1$$

and so

$$J(x) = \frac{1}{w - z} - \frac{1}{w - z} \sum_{n=0}^{\infty} (n + 1)\, a_n(x)\, d_n.$$

Consequently,

$$\lim_{n\to\infty} J(x) = \frac{1}{w - z}$$

provided that

$$\sum_{n=0}^{\infty} (n + 1)\, a_n(x)\, d_n = o(1)\ (x \to \infty). \qquad \ldots(2.6)$$

We now investigate the region, where eqn. (2.6) is satisfied. Using the results (2.1), (2.2) and (2.4) in (2.6), we have

$$\sum_{n=0}^{\infty} a_n(x)\,(n + 1)\, d_n$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} a_n(x)\,(n + 1) \int_0^{\pi} \int_0^{\infty} \left(\frac{\zeta}{\tau}\right)^n \left(\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right) d\phi\, d\nu \qquad \ldots(2.7)$$

It is not difficult to see (cf. King 1968) that

$$\int_0^{\pi} \int_0^{\infty} (| \zeta/\tau | + | 1/\tau |^{2}) \, d\phi \, d\nu$$

is uniformly bounded for $z \in R(w, \gamma, \beta)$ so that

$$\int_0^{\pi} \int_0^{\infty} (\zeta\tau^{-1} - \tau^{-2}) \, d\phi \, d\nu$$

is uniformly bounded over $R(w, \gamma, \beta)$. A formal interchange of integration and summation in (2.7) yields

$$\sum_{n=0}^{\infty} a_n(x) (n + 1) d_n$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_0^{\pi} (\zeta\tau^{-1} - \tau^{-2}) \sum_{n=0}^{\infty} a_n(x) (n + 1) (\zeta/\tau)^n \, d\phi \, d\nu \qquad \ldots(2.8)$$

From (1.1) and (1.2), we obtain

$$\sum_{n=0}^{\infty} a_n(x) (n + 1) (\zeta/\tau)^n$$

$$= \sum_{n=0}^{\infty} (n + 1) (\zeta/\tau)^n \int_0^{\infty} e^{-ux} \frac{(ux)^n}{n!} \, d\alpha(u). \qquad \ldots(2.9)$$

Applying Lebesgue's theorem (Titchmarsh 1968, p. 337; cf. Jakimovski 1960, p. 144; Leviatan and Lorch 1970, p. 68) for the integration of boundedly convergent series, (2.9) may be written as

$$\sum_{n=0}^{\infty} a_n(x) (n + 1) (\zeta/\tau)^n$$

$$= \int_0^{\infty} e^{-ux} \sum_{n=0}^{\infty} \frac{(ux(\zeta/\tau))^n}{n!} (n + 1) \, d\alpha(u)$$

$$= \int_0^{\infty} e^{-ux} [e^{ux(\zeta/\tau)}(1 + ux(\zeta/\tau))] \, d\alpha(u)$$

$$= \int_0^{\infty} e^{-ux(1-(\zeta/\tau))} (1 + ux(\zeta/\tau)) \, d\alpha(u)$$

(*equation continued on p.* 92)

$$= [e^{-ux(1-(\zeta/\tau))}(1 + ux(\zeta/\tau))\,\alpha(u)]_0^\infty$$

$$- \int_0^\infty e^{-ux(1-(\zeta/\tau))}\left[x\left(\frac{2\zeta}{\tau} - 1\right) - ux^2\left(\frac{\zeta}{\tau}\right)\left(1 - \frac{\zeta}{\tau}\right)\right]\alpha(u)\,du$$

$$= I_1 - I_2, \text{ say.} \qquad \ldots(2.10)$$

From (1.3), $\alpha(0) = 0$, so that the first term on the right side of (2.10) vanishes at the lower limit, while at the upper limit, it again becomes zero since, by hypothesis,

$$\text{Re}\,(\zeta/\tau) < \gamma < 1.$$

Hence $$I_1 = o(1) \qquad (x \to \infty). \qquad \ldots(2.11)$$

Since $\alpha(u)$ is of bounded variation, we may write

$$\alpha(u) = \lambda(u) - \mu(u)$$

where $\lambda(u)$ and $\mu(u)$ are both monotonic and bounded in u. It follows that

$$I_2 = \int_0^\infty e^{-ux(1-(\zeta/\tau))}\left[x\left(\frac{2\zeta}{\tau} - 1\right) - ux^2(\zeta/\tau)\left(1 - \frac{\zeta}{\tau}\right)\right]\lambda(u)\,du$$

$$- \int_0^\infty e^{-ux(1-(\zeta/\tau))}\left[x\left(\frac{2\zeta}{\tau} - 1\right) - ux^2(\zeta/\tau)\left(1 - \frac{\zeta}{\tau}\right)\right]\mu(u)\,du$$

$$= I_{2,1} - I_{2,2}, \text{ say.} \qquad \ldots(2.12)$$

Applying the second mean value theorem, we observe that

$$I_{2,1} = \lambda(0+)\int_0^\xi e^{-ux(1-(\zeta/\tau))}\left[x\left(\frac{2\zeta}{\tau} - 1\right) - ux^2(\zeta/\tau)\left(1 - \frac{\zeta}{\tau}\right)\right]du$$

$$+ \lambda(\infty-)\int_\xi^\infty e^{-ux(1-(\zeta/\tau))}\left[x\left(\frac{2\zeta}{\tau} - 1\right) - ux^2(\zeta/\tau)\left(1 - \frac{\zeta}{\tau}\right)\right]du$$

$$= I_{2,1,1} + I_{2,1,2}, \text{ say,} \qquad \ldots(2.13)$$

where $0 < \xi < \infty$. In like manner,

$$I_{2,2} = I_{2,2,1} + I_{2,2,2}, \qquad \ldots(2.14)$$

where $I_{2,2,1}$ and $I_{2,2,2}$ are expressions identical to those in (2.13), with $\lambda(0+)$, $\lambda(\infty-)$ and ξ, being replaced, respectively, by $\mu(0+)$, $\mu(\infty-)$ and ξ', $0 < \xi' < \infty$. Easy calculations now show that

$$I_{2,1,1} = \lambda(0+) \exp\{-x\xi(1-\zeta\tau^{-1})\}(1+\xi x\zeta\tau^{-1}) - \lambda(0+)$$

$$I_{2,2,1} = \mu(0+) \exp\{-x\xi'(1-\zeta\tau^{-1})\}(1+\xi' x\zeta\tau^{-1}) - \mu(0+)$$

$$I_{2,1,2} = \lambda(\infty-)\left[\exp\{-ux(1-\zeta\tau^{-1})\}(1+ux\zeta\tau^{-1})\right]_{\xi}^{\infty}$$

and

$$I_{2,2,2} = \mu(\infty-)\left[\exp\{-ux(1-\zeta\tau^{-1})\}(1+ux\zeta\tau^{-1})\right]_{\xi'}^{\infty}$$

and it could be verified that

$$I_{2,1,2} = o(1) \qquad (x \to \infty) \qquad \text{...(2.15)}$$

$$I_{2,2,2} = o(1) \qquad (x \to \infty). \qquad \text{...(2.16)}$$

Also

$$\begin{aligned} I_{2,1,1} - I_{2,2,1} &= \lambda(0+) \exp\{-x\xi(1-\zeta\tau^{-1})\}(1+\xi x\zeta\tau^{-1}) \\ &\quad -\mu(0+) \exp\{-x\xi'(1-\zeta\tau^{-1})\}(1+\xi' x\zeta\tau^{-1}) \\ &\quad - [\lambda(0+) - \mu(0+)] \end{aligned}$$

$$= o(1) - \alpha(0) = o(1) \qquad (x \to \infty). \qquad \text{...(2.17)}$$

It follows from (2.12) to (2.17) that

$$I_2 = o(1) \qquad (x \to \infty). \qquad \text{...(2.18)}$$

(2.10), (2.11) and (2.18) together imply that

$$\sum_{n=0}^{\infty} a_n(x)(n+1)(\zeta/\tau)^n = o(1) \qquad (x \to \infty) \qquad \text{...(2.19)}$$

which, when applied in (2.8), yields the required theorem.

The corresponding results for the Borel, classical Abel, and A_γ methods follow as particular cases from Theorem 2.1 by taking for $\alpha(u)$ the appropriate function given in section 1.

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References

Cowling, V. F., and King, J. P. (1962–63). On the Taylor and Lototsky summability of series of Legendre polynomials. *J. Analyse Math.*, **10**, 139–52.

Jakimovski, A. (1964). Analytic continuation and summability of series of Legendre polynomials. *Q. Jl Math.*, **15**, 289–302.

Jakimovski, A. (1966). Summability of the Heine and Neumann series of Legendre polynomials. *Canad. J. Math.*, **18**, 1261–63.

——— (1960). The sequence-to-function analogues to Hausdorff transformations. *Bull. Res. Council Israel*, pp. 135–54.

King, J. P. (1968). Some results for Borel transforms. *Proc. Am. math. Soc.*, **19**, 991–97.

Leviatan, D., and Lorch, L. (1970). The Gibbs phenomenon and Lebesgue constants for regular $[J, f(x)]$ means. *Acta Math. Acad. Sci. Hung.*, **21**, 65–85.

Powell, R. E. (1966). The $L(r, t)$ summability transform. *Canad. J. Math.*, **18**, 1251–60.

——— (1967). The $J(r_n)$ summability transform. *Canad. J. Math.*, **20**, 289–304.

Prachar, K. (1948–49). Zur Eulerschen Summierung Neumannscher and Legendrescher Reihen. *Monatsh. Math.*, **52–53**, 138–50.

Swaminathan, V. (1976). A note on the Valiron method of summability. *Acta Math. Acad. Sci. Hung.*, **28**, 211–17.

Titchmarsh, E. C. (1968). The Theory of Functions. Clarendon Press, Oxford.

Whittaker, E. T., and Watson, G. N. (1952). A Course of Modern Analysis. Cambridge University Press, London.