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ON INCLUSION RELATION BETWEEN ABSOLUTE RIESZ AND NÖRLUND MEANS

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McFadden's result and Hyslop's equivalence give the inclusion

$$\left| N, \frac{1}{n+1} \right| \subset | R, n+1, \alpha | \; \alpha > 0.$$

The author in the present paper establishes a general inclusion

$$| N, p_n | \subset | R, \lambda_{n-1}, 1 |$$

under two different sets of conditions on $\{p_n\}$ and $\{\lambda_n\}$ from which the above result and Das's inclusion $\left| N, \frac{1}{n+1} \right| \in | R, e^{(n-1)^\alpha}, 1 |$, $\alpha > 0$, follow.

§1. *Definitions and Notations* — Let $\{p_n\}$ be a sequence of constants real or complex, such that $P_n = p_0 + p_1 + \ldots + p_n \neq 0$ $(n \geqslant 0)$ and $P_n = p_n = 0$ $(n \leqslant -1)$.

Then the Nörlund transform of the sequence $\{s_n\}$ of partial sums of an infinite series $\sum_{n=0}^{\infty} a_n$ generated by the sequence $\{p_n\}$ is defined by

$$t_n(s_n) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v. \qquad \ldots(1.1)$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $| N, p_n |$ if $t_n(s_n) \in BV$ (Nörlund 1919).

We also require the Riesz method of summation. Let $\{\lambda_n\}$ be positive strictly increasing unbounded sequence. The Riesz mean or $(R, \lambda_{n-1}, 1)$ mean is defined by

$$R_n = \frac{1}{\lambda_n} \sum_{v=0}^{n} (\lambda_v - \lambda_{v-1}) s_v, \; (\lambda_{-1} = 0).$$

We say $\sum_{n=0}^{\infty} a_n$ is summable $| R, \lambda_{n-1}, 1 |$ (Obrechkoff 1929) if

$$\sum_{n=1}^{\infty} | R_n - R_{n-1} | < \infty. \qquad ...(1.2)$$

If every series summable by the method P is summable by the method Q, we write $P \subseteq Q$. If $P \subseteq Q$ and $Q \subseteq P$, then we say that the two methods are equivalent and we write $P \sim Q$.

We now formally define the sequence of constants $\{c_n\}$ by the identity

$$[\sum_{n=0}^{\infty} p_n x^n]^{-1} = \sum_{n=0}^{\infty} c_n x^n, \; c_{-1} = 0.$$

If, for $n = 0, 1, 2, ...$

$$p_n > 0, \; \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1.$$

We shall write $\{p_n\} \in \mathcal{M}$.

We write for any sequence $\{f_n\}$,

$$f_n^{(1)} = f_0 + f_1 + ... + f_n$$

$$f_n^{(2)} = f_0^{(1)} + ... + f_n^{(1)}.$$

We write $\Delta u_n = u_n - u_{n+1}$, $\bar{\Delta} u_n = u_n - u_{n-1}$.

§2. The following results are known:

$$| C, \alpha | \sim | R, n, \alpha |, \alpha \geqslant 0 \quad \text{(Hyslop 1937)} \qquad ...(2.1)$$

$$\left| N, \frac{1}{n+1} \right| \subset | C, \alpha |, \alpha > 0 \quad \text{(McFadden 1942)} \qquad ...(2.2)$$

$$\left| N, \frac{1}{n+1} \right| \underset{\not\supset}{\subset} | R, e^{(n-1)^\alpha}, 1 |, 0 < \alpha < 1 \quad \text{(Das 1969).} \qquad ...(2.3)$$

From (2.1) and (2.2) we also have

$$\left| N, \frac{1}{n+1} \right| \subset | R, n, \alpha | \sim | R, n+1, \alpha |. \qquad ...(2.4)$$

Our object is to prove more general inclusion theorems from which (2.3) and (2.4) follow.

§3. We prove the following theorems.

Theorem 1 — Let

$$\{p_n\} \in \mathcal{M}, \qquad ...(3.1)$$

$$\frac{P_m}{P_{m'}} = O(1)$$

where $m' = [m^{1-(\log\log\lambda_m/\log m)}] + 2 \quad (m' \to \infty)$ ...(3.2)

$$\frac{n}{\log \lambda_n} \frac{(\lambda_n - \lambda_{n-1})}{n} = O(1), \text{ as } n \to \infty \quad ...(3.3)$$

$$\frac{m' \log \lambda_m}{m} = O(1), \text{ as } m \to \infty. \quad ...(3.4)$$

Then $| N, p_n | \subset | R, \lambda_{n-1}, 1 |$.

Theorem 2 — If

$$\{p_n\} \in \mathcal{M} \quad ...(3.5)$$

$$P_m = O(P_{m'}), m' = [m^k] + 2, 0 < k < 1 \quad ...(3.6)$$

$\{\bar{\Delta} \lambda_n\}$ is positive non-decreasing and $n^k \bar{\Delta} \lambda_n = O(\lambda_n)$, then

$$| N, p_n | \subset | R, \lambda_{n-1}, 1 | . \quad ...(3.7)$$

Remarks : These results $| N, p_n | \subset | R, \lambda_{n-1}, 1 |$ follows from inclusion,

$$| N, p_n | \subset | R, \lambda_{n-1}, k | , k > 0 \quad ...(3.8)$$

(see Sukla 1979) by first theorem of consistency. We remark that the condition (3.7) of Theorem 2 is weaker than the analgous condition assumed in Sukla (1979). It is interesting to see if (3.8) is true under the restriction (3.7).

§4. For the proof of the theorems we require the following lemmas:

Lemma 1 (Das 1969) — Let $\{p_n\} \in \mathcal{M}$, then $\{t_n(s_n)\} \in BV$ iff

$$\sum_{n=1}^{\infty} \frac{| t_n(na_n) |}{n} < \infty.$$

Lemma 2 — Let $\{p_n\} \in \mathcal{M}$, then

(i) $c_0 > 0, c_n \leqslant 0$

(ii) $\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} p_n x^n)^{-1}$ is convergent for $| x | \leqslant 1$.

(iii) $\sum_{n=m+1}^{\infty} | c_n | \leqslant c_m^{(1)}$

(iv) $c_n^{(1)} \geqslant 0$ and is monotonic non-increasing

(v) $P_n c_n^{(1)} \leqslant 1$

(vi) $P_n c_n^{(2)} \leqslant (2n + 1)$.

For (i) to (iii) see Hardy (1949). For (iv), (v) and (vi) see Das (1969).

Lemma 3 — If (3.3), (3.4) are satisfied, then $\dfrac{\lambda_n}{\lambda_{n-1}} = O(1)$, as $n \to \infty$. This also follows from (3.7).

PROOF : From (3.3) and (3.4) we get

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \leqslant \frac{k \log \lambda_n}{n} = O(1).$$

This implies

$$\frac{\lambda_{n-1}}{\lambda_n} \to 1 \Rightarrow \frac{\lambda_n}{\lambda_{n-1}} \to 1 \text{ as } n \to \infty.$$

The proof of $\dfrac{\lambda_n}{\lambda_{n-1}} = O(1)$ by (3.7) is trivial.

Lemma 4 (Knopp and Lorentz 1949)—In order that the transformation $h = \sum_{v=0}^{n} \alpha_{nv} q_v$ should be absolutely regular it is necessary and sufficient that,

(i) $\sum_{v=0}^{n} \alpha_{nv} \to 1 \quad (n \to \infty)$

(ii) $\alpha_{nv} \to 0 \quad (n \to \infty,\ v \text{ fixed})$

(iii) $\sum_{n=m}^{\infty} \left| \sum_{v=m}^{\infty} (\alpha_{nv} - \alpha_{(n-1)v}) \right| \leqslant k$

uniformly in m. Condition (iii) is necessary and sufficient for $\{s_n\} \in BV$ to imply $\{h_n\} \in BV$.

Proof of Theorem 1

We have

$$R_n - R_{n-1} = \frac{\lambda_n - \lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \sum_{v=1}^{n} \lambda_{v-1} a_v.$$

Now expressing a_v in terms of $t_m(ma_m)$ we get

$$R_n - R_{n-1} = \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \sum_{m=1}^{n} P_m t_m(ma_m) \sum_{v=m}^{n} \frac{\lambda_{v-1}}{v} c_{v-m}.$$

To prove that $\sum_{n=1}^{\infty} | R_n - R_{n-1} | < \infty$ whenever $\sum_{m=1}^{n} \frac{| t_m(ma_m) |}{m} < \infty$. For the truth of this it is enough to show that

$$J_m = mP_m \sum_{n=m}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left| \sum_{v=m}^{n} \frac{\lambda_{v-1}}{v} c_{v-m} \right| = O(1).$$

Write $l = \min(n, m + m')$, where m' is as defined in (3.2). Then,

$$J_m \leqslant mP_m \sum_{n=m}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left| \sum_{v=m}^{l} \frac{\lambda_{v-1}}{v} c_{v-m} \right| + \sum_{n=m+m'+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}$$

$$\times \left| \sum_{v=m+m'+1}^{n} \frac{\lambda_{v-1}}{v} c_{(v-m)} \right|$$

$$= J_m^{(1)} + J_m^{(2)}$$

where

$$J_m^{(1)} = mP_m \sum_{n=m}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left| \sum_{v=m}^{l} \frac{\lambda_{v-1}}{v} c_{v-m} \right|$$

$$= mP_m \sum_{n=m}^{m+m'} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left| \sum_{v=m}^{l} \triangle \left(\frac{\lambda_{v-1}}{v} \right) c_{v-m}^{(1)} + \frac{\lambda_l}{l+1} c_{l-m}^{(1)} \right|$$

$$+ mP_m \sum_{n=m+m'+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left| \sum_{v=m}^{l} \triangle \left(\frac{\lambda_{v-1}}{v} \right) c_{v-m}^{(1)} + \frac{\lambda_l}{l+1} c_{l-m}^{(1)} \right|$$

$$\leqslant mP_m \sum_{n=m}^{m+m'} c_{v-m}^{(1)} \left| \triangle \left(\frac{\lambda_{v-1}}{v} \right) \right| \sum_{n=v}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}$$

$$+ mP_m \sum_{n=m}^{m+m'} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \frac{\lambda_n}{n+1} c_{n-m}^{(1)}$$

$$+ mP_m \frac{\lambda_{m+m'}}{m+m'} c_{m'}^{(1)} \sum_{n=m+m'+1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}$$

$$= J_{m,1}^{(1)} + J_{m,2}^{(1)} + J_{m,3}^{(1)}.$$

Now using the fact that $\sum_{n=\nu}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} = \frac{1}{\lambda_{\nu-1}}$, we get

$$
\begin{aligned}
J_{m,1}^{(1)} &\leqslant mP_m \sum_{\nu=m}^{m+m'} c_{\nu-m}^{(1)} \left| \Delta \left(\frac{\lambda_{\nu-1}}{\nu} \right) \right| \sum_{n=\nu}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \\
&\leqslant mP_m \sum_{\nu=m}^{m+m'} \frac{c_{\nu-m}^{(1)}}{\nu+1} (\lambda_\nu - \lambda_{\nu-1}) \frac{1}{\lambda_{\nu-1}} \\
&\quad + mP_m \sum_{\nu=m}^{m+m'} c_{\nu-m}^{(1)} \frac{\lambda_{\nu-1}}{\nu(\nu+1)} \frac{1}{\lambda_{\nu-1}} \\
&= O(1)\, mP_m \sum_{\nu=m}^{m+m'} c_{\nu-m}^{(1)} \frac{\lambda_\nu}{(\nu+1)} \frac{\log \lambda_\nu}{\nu \lambda_{\nu-1}} \\
&\quad + O(1)\, mP_m \sum_{\nu=m}^{m+m'} c_{\nu-m}^{(1)}/\nu^2 \qquad \text{(by 3.3)} \\
&= O(1) \frac{mP_m}{m^2} \log \lambda_m \frac{2m'+1}{P_{m'}} + O(1) \frac{P_m}{m} \frac{2m'+1}{P_{m'}} \\
&= O(1),
\end{aligned}
$$

by Lemma 2(vi), (3.2) and (3.4).

Next,

$$
\begin{aligned}
J_{m,2}^{(1)} &= O(1) \left[mP_m \sum_{n=m}^{m+m'} c_{n-m}^{(1)} \frac{\log \lambda_n}{n^2} \right] \\
&= O(1) \left[mP_m \sum_{n=m}^{m+m'} c_{n-m}^{(1)} \frac{1}{nn'} \right] \\
&= O\left[\frac{P_m}{m'} \frac{m'}{P_{m'}} \right] = O(1)
\end{aligned}
$$

by Lemma 2(vi), (3.4) and (3.2).

Next,

$$
J_{m,3}^{(1)} = O(1) \frac{mP_m}{m+m'} \lambda_{m+m'} c_{m'}^{(1)} \frac{1}{\lambda_{m+m'}}
$$

(*equation continued on p.* 101)

$$= O(1),$$

by Lemma 2(v) and (3.2).

Lastly,

$$J_m^{(2)} \leqslant mP_m \sum_{v=m+m'+1}^{\infty} \frac{\lambda_{v-1}}{v} \mid c_{v-m} \mid \sum_{n=v}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}$$

$$\leqslant mP_m \sum_{v=m+m'+1}^{\infty} \frac{\mid c_{v-m} \mid}{v} = O(1),$$

by Lemma 2(iii), (v) and (3.2).

This completes the proof of Theorem 1.

Proof of Theorem 2

Proceeding as in the proof of Theorem 1,

$$J_m \leqslant J_m^{(1)} + J_m^{(2)} \text{ and } J_m^{(1)} \leqslant J_{m,1}^{(1)} + J_{m,2}^{(1)} + J_{m,3}^{(1)}.$$

In place of (3.3) we use here (3.7) and have

$$J_{m,1}^{(1)} = O(1)\, mP_m \sum_{v=m}^{m+m'} c_{v-m}^{(1)} \frac{\lambda_v}{v^k(v+1)} \frac{1}{\lambda_{v-1}}$$

$$+ O(1)\, mP_m \sum_{v=m}^{m+m'} \frac{c_{v-m}^{(1)}}{v(v+1)}$$

$$= O(1) \frac{mP_m}{m^{k+1}} \frac{(2m'+1)}{P_{m'}} + O(1) \frac{P_m}{m} \frac{(2m'+1)}{P_{m'}}$$

$$= O(1), \quad \text{by lemma 2(vi) and (3.6).}$$

Similarly,

$$J_{m,2}^{(1)} = O(1)\, mP_m \sum_{n=m}^{m+m'} \frac{c_{n-m}^{(1)}}{n^k(n+1)} = O(1).$$

Also

$$J_{m,3}^{(1)} = O(1) \text{ and } J_m^{(2)} = O(1),$$

by the same argument as in the proof of Theorem 1.

Hence the theorem is proved.

Corollary 1 — $\left| N, \frac{1}{n+1} \right| \subset | R, e^{(n-1)^\alpha}, 1 |$, $0 < \alpha < 1$ follows by taking $p_n = \frac{1}{n+1}$, $\lambda_{n-1} = e^{(n-1)^\alpha}$, $0 < \alpha < 1$ in Theorem 1 or by taking $k = 1 - \alpha (0 < \alpha < 1)$, $p_n = \frac{1}{n+1}$, $\lambda_{n-1} = e^{(n-1)^\alpha}$ in Theorem 2, since the definition $| R, \lambda_{n-1}, 1 |$ is not affected by the change of any finite number of λ_n's.

Corollary 2 — $\left| N, \frac{1}{n+1} \right| \subset | R, n, 1 |$ follows by taking $p_n = \frac{1}{n+1}$, $\lambda_n = n+1$ in Theorem 2.

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