Indian J. pure appl. Math., 12(8) : 1001–1006, August 1981

SOME RESULTS CONCERNING A SPECIAL FUNCTION OF SEVERAL COMPLEX VARIABLES

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(*Received* 8 *September* 1980)

The object of the present paper is to evaluate certain finite integrals associated with the multiple H-function, defined recently by Srivastava and Panda (1976). The results are most general in nature and are belived to be new. Both new and known results, involving certain special functions in one and more arguments, may be easily derived as their particular or limiting cases.

1. INTRODUCTION

Recently, Srivastava and Panda (1976) have defined the H-function of several complex variables, which incorporates (as its special cases) the special H-function of n variables defined by Saxena (1974), the G-function of n variables given earlier by Khadia and Goyal (1970), the H-function of two variables defined by various authors scattered in the literature, and the H-function of Fox. This function also generalizes nearly all the known special functions of n variables, e.g., Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$, etc. Besides including the known special functions of n variables, at its particular cases, it leaves the possibility of defining through this new H symbol, a great many special functions of n variables not so far recorded in the literature. Very recently, Buschman (1977, 1978) has critically discussed the convergence conditions for the H-function of two and more variables.

Throughout this paper we follow Srivastava and coworkers (1976, 1978) employ the abbreviation (a) to denote the sequence of A parameters $a_1, \ldots, a_A$; for each $i = 1, \ldots, n$, $(b^{(i)})$ abbreviates the sequences of $B^{(i)}$ parameters $b_j^{(i)}$, $j = 1, \ldots, B^{(i)}$, with similar interpretations for (c), $(d^{(i)})$, etc., $i = 1, \ldots, n$; it would be understood, for example, that $b^{(1)} = b'$, $b^{(2)} = b''$, and so on. Also for the sake of brevity, we use the contracted notations

$$[(a)]_n = \prod_{j=1}^{A} [a_j]_n,\ [(b^{(i)})]_n = \prod_{j=1}^{B^{(i)}} [b_j^{(i)}]_n,\ i = 1, \ldots, n, \text{ etc.} \qquad \ldots(1.1)$$

The multiple H-function given by Srivastava and Panda (1976) shall be defined and represented by means of the multiple contour integral as follows [Srivastva and Panda 1976, p. 271, eqn. (4.1) *et. seq.*; see also Srivastava and Srivastava 1978, pp. 59–60]:

$$H[z_1, \ldots, z_n] = H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix} [(a):\theta', \ldots, \theta^{(n)}]:[(b'),\phi'];\ldots;[(b^{(n)}),\phi^{(n)}]; \\ [(c):\psi', \ldots, \psi^{(n)}]:[(d'),\delta'];\ldots;[(d^{(n)}),\delta^{(n)}]; \end{matrix} \; z_1, \ldots, z_n\right)$$

$$= (2\pi w)^{-n} \int_{-w\infty}^{w\infty} \ldots \int_{-w\infty}^{w\infty} \phi_1(s_1) \ldots \phi_n(s_n)\, \psi(s_1, \ldots, s_n)$$

$$\times\, z_1^{(s_1)} \ldots z_n^{(s_n)}\, ds_1 \ldots ds_n, \quad w = \sqrt{-1} \qquad \ldots(1.2)$$

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma(d_j^{(i)} - s_i\delta_j^{(i)}) \prod_{j=1}^{\nu^{(i)}} \Gamma[1 - b_j^{(i)} + s_i\phi_j^{(i)}]}{\prod_{j=1+\mu^{(i)}}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + s_i\delta_j^{(i)}) \prod_{j=1+\nu^{(i)}}^{B^{(i)}} \Gamma(b_j^{(i)} - s_i\phi_j^{(i)})}, \qquad \ldots(1.3)$$

$$\forall i \in \{1, \ldots, n\}.$$

$$\psi(s_1, \ldots, s_n) = \frac{\prod_{j=1}^{\lambda} \Gamma[1 - a_j + \sum_{i=1}^{n} s_i\,\theta_j^{(i)}]}{\prod_{j=\lambda+1}^{A} \Gamma[a_j - \sum_{i=1}^{n} s_i\,\theta_j^{(i)}] \prod_{j=1}^{C} \Gamma[1 - c_j + \sum_{i=1}^{n} s_i\psi_j^{(i)}]} \qquad \ldots(1.4)$$

an empty product is interpreted as unity, the coefficients $\theta_j^{(i)}$, $j = 1, \ldots, A$; $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\psi_j^{(i)}, j = 1, \ldots, C$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$; $\forall i \in \{1, \ldots, n\}$, are positive numbers, and λ, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are integers such that

$$0 \leqslant \lambda \leqslant A,\ 0 \leqslant \mu^{(i)} \leqslant D^{(i)},\ C \geqslant 0, \text{ and } 0 \leqslant \nu^{(i)} \leqslant B^{(i)};$$

$\forall i \in \{1, \ldots, n\}$. The paths of integration are indented, if necessary, in such a way that all the poles of $\Gamma[d_j^{(i)} - s_i\delta_j^{(i)}]$, $j = 1, \ldots, \mu^{(i)}$, are separated from the poles of

$$\Gamma[1 - b_j^{(i)} + s_i\phi_j^{(i)}],\ j = 1, \ldots, \nu^{(i)},$$

and $$\Gamma[1 - a_j + \sum_{i=1}^{n} s_i\theta_j^{(i)}],\ j = 1, \ldots, \lambda.$$

When $z_1, \ldots, z_n$ are not equal to zero, the multiple integral in (1.2) converges absolutely if

$$|\arg z_i| < \tfrac{1}{2}\pi\Delta_i, \quad \forall i \in \{1, \ldots, n\}, \qquad \ldots(1.5)$$

where

$$\Delta_i = \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \quad \forall i \in \{1, \ldots, n\}. \qquad \ldots(1.6)$$

The conditions corresponding to the aforementioned ones shall be assumed to hold good throughout this paper. The author obtained a number of results involving the H-function of two and more variables (see Mathur 1975, 1976, 1979a, b, c; Mathur and Krishna 1978; Munot and Mathur 1975, 1976, 1978). Here we give some more interesting results associated with the (Srivastava-Panda) H-function of several complex variables defined by (1.2).

2. Evaluation of the Integrals

The first integral to be evaluated is

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}Z^{-\alpha-\beta} H\left[\left(\frac{x}{Z}\right)^{m_1}\left(\frac{1-x}{Z}\right)^{\rho_1} u_1, \ldots, \left(\frac{x}{Z}\right)^{m_n}\left(\frac{1-x}{Z}\right)^{\rho_n} u_n\right] dx$$

$$= (1+\xi)^{-\alpha}(1+\eta)^{-\beta} H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\ldots;[B^{(n)},D^{(n)}]} \left(\begin{matrix} [(1-\alpha):m_1,\ldots,m_n],[(1-\beta):\rho_1,\ldots,\rho_n],[(a):\theta',\ldots,\theta^{(n)}]: \\ [(1-\alpha-\beta):m_1+\rho_1,\ldots,m_n+\rho_n],[(c):\psi',\ldots,\psi^{(n)}]: \end{matrix} \begin{matrix} [(b'),\phi'];\ldots;[(b^{(n)}),\phi^{(n)}]; \\ [(d'),\delta'];\ldots;[(d^{(n)}),\delta^{(n)}]; \end{matrix}\; u_1R_1,\ldots,u_nR_n\right) \qquad \ldots(2.1)$$

provided

$$\text{Re}\,[\alpha + \sum_{k=1}^{n} m_k\sigma_k] > 0,\ \text{Re}\,[\beta + \sum_{k=1}^{n} \rho_k\sigma_k] > 0, \quad \sigma_k = d_j^{(k)}/\delta_j^{(k)},$$

$(k = 1, \ldots, n\,; j = 1, \ldots, \mu^{(i)}), m_k > 0; \rho_k > 0\,(k = 1, \ldots, n)$, $|\arg(u_i)| < \frac{1}{2}\pi\Delta_i,\ \Delta_i > 0$, $Z = [1 + \xi x + \eta(1-x)]$, ξ and η are such that none of the expressions $1+\xi$, $1+\eta$, $[1+\xi x+\eta(1-x)]\ (0 \leqslant x \leqslant 1)$ is zero and $R_k = (1+\xi)^{-m_k}(1+\eta)^{-\rho_k}$, PROOF: In the integrand of (2.1) we replace the multiple H-function by its Mellin-Barnes contour integral (1.2) and invert the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, and evaluate the x-integral with the help of a known result [MacRobert 1961, p. 451] and finally interpret the result with (1.2); we readily arrive at the desired result.

Proceeding in a similar way and using the known relations (Mathai and Saxena 1973, pp. 73, 71) we obtain the following two results respectively:

$$\int_0^1 x^{\xi}(1-x)^{\xi-1}\, {}_2F_1[\alpha+\beta;\ (\alpha+\beta+1)/2;\ x]\ H[x^{m_1}(1-x)^{m_1} \times u_1, \ldots, x^{m_n}(1-x)^{m_n}u_n]\, dx$$

$$= \pi 2^{1-2\xi}\, \Gamma((\alpha+\beta+1)/2)\, [\Gamma((1+\alpha)/2)\, \Gamma((1+\beta)/2)]^{-1} \times H^{0,\lambda+2:(\mu',v');\ldots;(\mu^{(n)},v^{(n)})}_{A+2,C+2:[B',D'];\cdots;[B^{(n)},D^{(n)}]} \left(\begin{matrix} [(1-\xi):m_1,\ldots,m_n], \\ [(1+\alpha-2\xi)/2:m_1,\ldots,m_n], \end{matrix} \right.$$

$$\begin{matrix} [(\alpha+\beta+1-2\xi)/2:m_1,\ldots,m_n],[(a):\theta',\ldots,\theta^{(n)}]: \\ [(1+\beta-2\xi)/2:m_1,\ldots,m_n],[(c):\psi',\ldots,\psi^{(n)}] \end{matrix}$$

$$\left. \begin{matrix} [(b'),\phi'];\ldots;[(b^{(n)}),\phi^{(n)}]; \\ [(d'),\delta'];\ldots;[(d^{(n)}),\delta^{(n)}]; \end{matrix}\ 4^{-m_1}u_1,\ldots,4^{-m_n}u_n \right) \quad \ldots(2.2)$$

provided

$$\text{Re}\,[\xi + \sum_{k=1}^{n} m_k\sigma_k] > 0,\ \text{Re}\,[(1-\alpha-\beta+2\xi)/2 + \sum_{k=1}^{n} m_k\sigma_k] > 0;$$

$$m_k > 0,\ \sigma_k = d_j^{(k)}/\delta_j^{(k)}\ (j=1, =, \mu^{(i)};\ k=1,\ldots,n),$$

$$|\arg(u_i)| < \tfrac{1}{2}\pi\Delta_i,\ \Delta_i > 0.$$

$$\int_0^{\pi/2} (\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1} e^{i(\alpha+\beta)\theta}\, {}_2F_1(\xi,\eta;\beta;e^{i\theta}\cos\theta) \times H[(\sin\theta)^{t_1}(\cos\theta)^{r_1} e^{i(r_1+t_1)\theta}u_1,\ldots,(\sin\theta)^{t_n}(\cos\theta)^{r_n} \times e^{i(r_n+t_n)\theta}u_n]\, d\theta$$

$$= e^{(i\pi\alpha)/2}\, H^{0,\lambda+3:(\mu'v');\ldots;(\mu^{(n)},v^{(n)})}_{A+3,C+2:[B',D'];\cdots;]B^{(n)},D^{(n)}]} \left(\begin{matrix} [(1-\alpha):t_1,\ldots,t_n), \\ [(1-\alpha-\beta+\xi):t_1,\ldots,t_n], \end{matrix} \right.$$

(equation continued on p. 1005)

$$[(1-\beta):r_1,\ldots,r_n],[1-\alpha-\beta+\xi+\eta:r_1+t_1,\ldots,r_n+t_n],$$
$$[(1-\alpha-\beta+\eta):r_1,\ldots,r_n],$$
$$\left.\begin{array}{l}[(a):\theta',\ldots,\theta^{(n)}]:[(b'),\phi'];\ldots;[(b^{(n)}),\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'),\delta'];\ldots;[(d^{(n)}),\delta^{(n)}];\end{array} e^{i\pi/2t_1}u_1,\ldots,e^{i\pi/2t_n}u_n\right) \quad \ldots(2.3)$$

provided $\operatorname{Re}[\alpha+\sum_{k=1}^{n} t_k\sigma_k]>0$, $\operatorname{Re}[\beta+\sum_{k=1}^{n} r_k\sigma_k]>0$,

$$\operatorname{Re}[\alpha+\beta-\xi-\eta+\sum_{k=1}^{n}(t_k+r_k)\sigma_k]>0$$

$$(\sigma_k=d_j^{(k)}/\delta_j^{(k)};\ k=1,\ldots,n,\ j=1,\ldots,\mu^{(i)}),\ r_k>0,\ t_k>0$$

$$(k=1,\ldots,n),\ |\arg(u_i)|<\tfrac{1}{2}\pi\Delta_i,\ \Delta_i>0.$$

3. Applications

When all the θ's, ϕ's, ψ's and δ's are choosen to be 1, the multiple H-function defined by (1.2) would evidently reduce to the corresponding G-function of n variables, which in essence is same as given by Khadia and Goyal (1970). The results involving this function may thus be derived easily from (2.1) – (2.3).

For $\theta_j'=\ldots=\theta_j^{(n)}, j=1,\ldots,A$ and $\psi_j'=\ldots=\psi_j^{(n)}, j=1,\ldots,C$, the multiple H-function (1.2) would reduce to the special. H-function of n variables considered recently by Saxena (1974, p. 255).

If $\lambda=A=C=0$, the second member of (1.2) would evidently degenerate itself into the product of n mutually independent Mellin-Barnes contour integrals, each representing a distinct H-function of one variable. Thus the results represented in this paper can first be reduced fairly easily to hold for the product of several H-functions {and hence also G-functions, E-functions, F-functions or Wright's generalized hypergeometric functions, and so on} of different arguments. In particular, by taking $n=2$ in (2.1) – (2.3) we easily derive results involving H-function of two variables. Similarly, the results obtained earlier by Mathur (1970) for the Fox's H-function are the particular case of (2.1) – (2.3).

It may be recalled that a great many of special functions that occur in the problems of applied mathematics and mathematical physics can be expressed in terms of the G-function, and hence also as a H-function. Evidently, therefore, our results (2.1) – (2.3) would reduce to the simpler special functions of mathematical physics, such as the Legendre functions, the Bessel functions, the Whittaker functions, the classical orthogonal polynomials of Jacobi, Laguerre and Hermite, etc. By making a free use of the tables (Mathai and Saxena 1973), one can easily obtain several interesting applications of our results.

ACKNOWLEDGEMENT

Thanks are due to Sri A. Nagaratnam and Dr S. Krishna for giving inspiration and for providing necessary facilities. Thanks are also due to Prof. H. M. Srivastava of the University of Victoria, Canada, for encouragement and also for providing several references including, for example, Srivastava and Panda (1976), and Srivastava and Srivastava (1978).

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