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VISCOUS IMPULSIVE ROTATION OF TWO FINITE COAXIAL DISKS

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The axisymmetric viscous agitation due to slow impulsive arbitrary rotations of two finite coaxial circular disks in an unbounded incompressible fluid is studied, using suitable numerical algorithms. Torques on the disks are calculated. The influence due to the presence of the second disk on the flow field is discussed. Allowing $t \to \infty$, steady-state analysis is realised. The behaviour of the corresponding swirl along the axis of rotation is analysed. Disk friction losses due to relatively slow rotation (same sense) of the second disk are observed. Corresponding analysis for a single disk in (i) infinite and (ii) semi-infinite medium are deduced as particular cases.

1. INTRODUCTION

It is well-known that the mixed boundary value problems reduce to solving dual or triple integral equations (Sneddon 1966). Cooke (1956, 1958) has studied the axisymmetric slow rotation of two finite coaxial disks in an unbounded medium and calculated the torque on each disk, when h (separation distance) $\geqslant 0.4$. As an alternative, he proposed an asymptotic approximation to the turning couple, which was later improved by Hutson (1964). Subsequently Susann-Shaw (1970) used the method of matched asymptotic expansions to the same problem of counter-rotating disks when $h \leqslant 1$, taking into consideration the thickness of the disks. All the above problems deal with steady-state solutions. Slow viscous rotatory-oscillations of a circular disk has been studied by Kanwal (1970).

In this paper the problem of viscous agitation, due to slow unsteady rotations about the axis of symmetry of two coaxial disks $0 \leqslant R \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, $Z = 0$ or h, with arbitrary impulsive angular velocities $\Omega_0(T)$ and $\Omega_h(T)$, in a large expanse of otherwise undisturbed fluid, is studied. Secondary flows are neglected. With the use of Laplace transforms and dual integral equation techniques, the problem reduces to solving a pair of Fredholm's integral equations of second kind, which are numerically integrated and inverted simultaneously by adopting suitable algorithms. The torques on the disks are calculated for the cases in which

$$\text{(i) } \Omega_0(T) = 0.5\, H(T) \text{ or } 0.5 \exp(-T)\, H(T),\ \Omega_h(T) = 0$$

and

$$\text{(ii) } \Omega_0(T) = -\Omega_h(T) = 0.5\, H(T) \text{ or } 0.5 \exp(-T)\, H(T).$$

Lower bound for h is given as a function of time, for which the presence of the stationary disk has little influence over the rotating disk. The steady-state behaviour of swirl along the axis of rotation and torques on the disks are studied as limiting cases. When $0 < \Omega_h < \Omega_0$, it is observed that the rotation of the upper disk causes the torque on the faster rotating disk to fall even below the level of the corresponding single disk torque, depending on the separation parameter h.

The corresponding analysis for the unsteady rotation of a single disk in (i) infinite and (ii) semi-infinite medium bounded by a rigid wall, are deduced as particular cases. In the former case, an asymptotic expansion for the torque is given for large time.

2. Analysis

Equations of Motion and Solutions

Neglecting secondary flows and using Laplace transforms, the equation for the azimuthal velocity $V(R, Z, T)$ in terms of non-dimensional variables,

$$R = R'/a,\ Z = Z'/a,\ T = \nu T'/a^2,\ V = V'/a\omega_0,\ \Omega = \Omega'/\omega_0$$

(prime indicates physical variables, ω_0 a characteristic angular velocity), reduces to

$$p\bar{V}(R, Z, p) = \frac{\partial^2 \bar{V}}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{V}}{\partial R} - \frac{\bar{V}}{R^2} + \frac{\partial^2 \bar{V}}{\partial Z^2} \qquad ...(2.1)$$

subject to the boudary conditions

$$\bar{V}(R, Z, p) = \begin{cases} R\bar{\Omega}_0(p)\,;\ 0 \leqslant R \leqslant 1,\ Z = 0 \\ R\bar{\Omega}_h(p);\ 0 \leqslant R \leqslant 1,\ Z = h \end{cases} \qquad ...(2.2)$$

$$\frac{\partial \bar{V}_+}{\partial Z} - \frac{\partial \bar{V}_-}{\partial Z} = 0;\ R > 1,\ Z = 0 \text{ or } h \qquad ...(2.3)$$

$$\bar{V}(R, Z, p) \to 0 \text{ as } R \to \infty \text{ or } |Z| \to \infty \qquad ...(2.4)$$

$$\bar{V}(0, Z, p) = 0 \text{ for all } Z \qquad ...(2.5)$$

where $\overline{(\ )} = \int_0^\infty \exp(-pT)\,(\ )\,dT.$

An elementary solution satisfying (2.4) and (2.5) is given by

$$\bar{V}(R, Z, p) = \int_0^\infty \xi^{-1} \left[\begin{array}{l} A_1(\xi, p) \exp[-(\xi^2 + p)^{1/2} \mid Z \mid] \\ + A_2(\xi, p) \exp[-(\xi^2 + p)^{1/2} \mid Z - h \mid] \end{array} \right] \times J_1(\xi R)\, d\xi. \quad \ldots(2.6)$$

Setting

$$A_{1,2}(\xi, p) = (\tfrac{1}{2}) \frac{\xi}{(\xi^2 + p)^{1/2}} [P_1(\xi, p) \pm P_2(\xi, p)] \quad \ldots(2.7)$$

the conditions (2.2) and (2.3) determine $P_{1,2}(\xi, p)$ as solutions of

$$\int_0^\infty \xi^{-1} [1 + k_{1,2}(\xi, p)]\, P_{1,2}(\xi, p)\, J_1(\xi R)\, d\xi = R[\bar{\Omega}_0(p) \pm \bar{\Omega}_h(p)]; \quad 0 \leqslant R < 1 \quad \ldots(2.8)$$

$$\int_0^\infty P_{1,2}(\xi, p)\, J_1(\xi R)\, d\xi = 0; \; R > 1 \quad \ldots(2.9)$$

where

$$k_{1,2}(\xi, p) = \left(\frac{\xi}{(\xi^2 + p)^{1/2}} - 1 \right) \pm \frac{\xi}{(\xi^2 + p)^{1/2}} \exp[-(\xi^2 + p)^{1/2} h]. \quad \ldots(2.10)$$

Following Sneddon (1966), the above systems of dual integral equations give

$$P_{1,2}(\xi, p) = 2^{-1/2} \xi^{3/2} \int_0^1 R^{1/2} H_{1,2}(R, p)\, J_{1/2}(\xi R)\, dR \quad \ldots(2.11)$$

$H_{1,2}(R, p)$ are the respective solutions of

$$H_{1,2}(R, p) = (4/\pi^{1/2}) [\bar{\Omega}_0(p) \pm \bar{\Omega}_h(p)] R - \int_0^1 E_1(R, u, p) \pm E_2(R, u, p)]\, H_{1,2}(u, p)\, du \quad \ldots(2.12)$$

where

$$E_1(R, u, p) = (Ru)^{1/2} \int_0^\infty \sigma \left[\frac{\sigma}{(\sigma^2 + p)^{1/2}} - 1 \right] J_{1/2}(\sigma R) J_{1/2}(\sigma u)\, d\sigma \quad \ldots(2.13)$$

and

$$E_2(R, u, p) = (Ru)^{1/2} \int_0^\infty \frac{\sigma^2}{(\sigma^2 + p)^{1/2}} \exp[-(\sigma^2 + p)^{1/2} h] \times J_{1/2}(\sigma R)\, J_{1/2}(\sigma u)\, d\sigma. \quad \ldots(2.14)$$

Choosing $\varphi_j(s) = s\left(\frac{s}{(s^2+p)^{1/2}} - 1\right) H_\beta^{(j)}(su)\, J_\beta(sR) \quad (j = 1, 2)$

where $s = \sigma + i\tau$, u, R are real, H_β are Hankel functions, and integrating $\varphi_1(s)$ and $\varphi_2(s)$ respectively around the circular quadrants $C_1(\sigma > 0, \tau > 0)$, $C_2(\sigma > 0, \tau < 0)$ indented at the branch points $s = 0$, $s = \pm i p^{1/2}$, one obtains

$$E_1(R, u, p) = (\tfrac{1}{2})\, p^{1/2}\Big[\, [L_1(p^{1/2}(R+u)) - I_1(p^{1/2}(R+u))] - [L_1(p^{1/2}\,|R-u|\,) - I_1(p^{1/2}\,|R-u|\,)]\Big] \quad \text{...(2.15)}$$

where $L_1(x)$ and $I_1(x)$ are modified Struve function and modified Bessel function of first kind respectively.

Expressing $J_{1/2}(\sigma R)$, $J_{1/2}(\sigma u)$ in powers of σ and interchanging the order of integration and summation, $E_2(R, u, p)$ becomes

$$E_2(R, u, p) = (1/\pi^{3/2}) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{(2n)!}\Big[(u+R)^{2n} - (u-R)^{2n}\Big] \times (2p^{1/2}/h)^{(2n+1)/2} K_{(2n+1)/2}(hp^{1/2}). \quad \text{...(2.16)}$$

Expanding (2.16) in powers of $1/h$ $(h > 1)$ one obtains

$$\begin{aligned} E_2(R, u, p) = (1/\pi) \exp(-hp^{1/2})[[g(0,1)](1/h) + [g(0,2) + g(1,1)](1/h)^2 \\ + [g(0,3) + g(1,2)](1/h)^3 + [g(0,4) + g(1,3) + g(2,2)](1/h)^4 \\ + [g(0,5) + g(1,4) + g(2,3)](1/h)^5 + \ldots] \end{aligned} \quad \text{...(2.17)}$$

where

$$g(k, j-k) = (-1)^{j-k-1} \frac{[j-k]!}{[2(j-k)]!} \frac{[j]!}{[j-2k]!\,[k]!} \times [(u+R)^{2(j-k)} - (u-R)^{2(j-k)}]\,(2p^{1/2})^{j-2k}. \quad \text{...(2.18)}$$

Knowing $H_i(R, p)$; $P_i(\xi, p)$ $(i = 1, 2)$ can be determined from (2.11). The velocity field is then obtained from (2.6) and (2.7) subject to a successful Laplace inversion.

Torque

The torques on the disks non-dimensionalised by $\mu\omega_0 a^3$ are

$$M_{0,h}(T) = -2\pi \int_0^1 \left(\frac{\partial V_+}{\partial Z} - \frac{\partial V_-}{\partial Z}\right)_{z=0,h} R^2\, dR \quad \text{...(2.19)}$$

whose transformed expressions are given as

$$\bar{M}_{0,h}(p) = 4\pi \int_0^1 R^2 \int_0^\infty \xi^{-1}(\xi^2 + p)^{1/2} A_{1,2}(\xi, p) J_1(\xi R)\, d\xi\, dR. \qquad \ldots(2.20)$$

Further simplification leads to

$$\bar{M}_{0,h}(p) = 4\pi^{1/2} \int_0^1 \sigma[H_1(\sigma, p) \pm H_2(\sigma, p)]\, d\sigma. \qquad \ldots(2.21)$$

3. Numerical Computations

Gaussian quadrature formula (Abramowitz and Stegun 1968) and a numerical algorithm for Laplace inversion (Schapery 1962) are used to solve (2.12) for $H_{1,2}(R, p)$. Schapery's algorithm—Direct method—holds if $\frac{d[p\bar{f}(p)]}{d[\log_{10} p]}$ varies slowly as a function of $\log_{10} p$. This condition has been verified in the present context. To calculate $L_1(x)-I_1(x)$ at the required abcissae of the quadrature formula, series expansion—Watson (1966) is used. E_2 is evaluated by using the series (2.17) when $h \geqslant 5$ and directly evaluated by using Gauss-Laguerre quadrature formula for (2.14) when $0.07 \leqslant h < 5$. Use of ten point Gauss quadrature formula is found sufficient to compute the results for $0.07 \leqslant h \leqslant 100.0$, correct up to four decimal places.

4. Steady-State Analysis

Setting $\Omega_0(T) = 0.5\ H(T)$, $\Omega_h(T) = 0$ and applying final value theorem on Laplace transforms to the governing equations (2.6) - (2.14), the steady-state torque on the disks, correct up to $O(1/h^8)$ are obtained as

$$M_0(\infty) = (16/3) + (256/27\,\pi^2)(1/h^6) - (2048/45\,\pi^2)(1/h^8) - \ldots \qquad \ldots(4.1)$$

$$M_h(\infty) = -(64/9\,\pi)(1/h^3) + (256/15\,\pi)(1/h^5) - (1536/35\,\pi)(1/h^7) + \ldots \qquad \ldots(4.2)$$

As $h \to \infty$, corresponding torque on a single disk is realised (Jeffrey 1915). The steady-state swirl along the axis of rotation is defined as

$$G(Z) = \lim_{R \to 0} \frac{V(R, Z, \infty)}{R} \qquad \ldots(4.3)$$

whose asymptotic expansion when $h \gg 1$ is

$$G(Z) = G_\infty(Z) - (4/3\pi)\, G_\infty(Z - h)(1/h^3) + O(1/h^5) \qquad \ldots(4.4)$$

where $$G_\infty(Z) = (1/\pi)[\tan^{-1}(1/|Z|) - (|Z|/(1 + Z^2))] \qquad \ldots(4.5)$$

is the corresponding swirl due to a single disk rotation. It is observed that $M_0(\infty)$ decreases as $O(1/h^6)$ whereas $G(Z)$ increases as $O(1/h^3)$.

5. Single Disk in Infinite and Semi-infinite Medium

Infinite Medium

This situation is derived by setting $\Omega_h(T) = 0$ and allowing $h \to \infty$. The velocity field is obtained as

$$\bar{V}(R, Z, p) = (1/\pi^{1/2}) \int_0^1 H(\sigma, p) \int_0^\infty \frac{\xi}{(\xi^2 + p)^{1/2}} \exp[-(\xi^2 + p)^{1/2} \mid Z \mid] \times J_1(\xi R) \sin(\xi\sigma)\, d\xi\, d\sigma \qquad \ldots(5.1)$$

where H satisfies the equation

$$H(R, p) = (4/\pi^{1/2})\, \bar{\Omega}(p)\, R - \int_0^1 E_1(R, u, p)\, H(u, p)\, du. \qquad \ldots(5.2)$$

The torque is

$$\bar{M}_\infty(p) = 8\pi^{1/2} \int_0^1 \sigma H(\sigma, p)\, d\sigma. \qquad \ldots(5.3)$$

To obtain a solution valid for large time, writing

$$pH(\sigma, p) = \sum_{n=0}^{\infty} (p^{1/2})^n H_n(\sigma) \qquad \ldots(5.4)$$

$$p\bar{\Omega}(p) = \sum_{n=0}^{\infty} (p^{1/2})^n \Omega_n \qquad \ldots(5.5)$$

$$\begin{aligned} E_1(R, u, p) = {} & (1/4)[\,|R - u| - (R + u)]\,(p^{1/2})^2 + (4/3\,\pi)\, Ru\,(p^{1/2})^3 \\ & + (1/32)[\,|R - u|^3 - (R + u)^3]\,(p^{1/2})^4 \\ & + (8/45\,\pi)\, R(R^2u + u^3)\,(p^{1/2})^5 + (1/768)[\,|R - u|^5 \\ & - (R + u)^5]\,(p^{1/2})^6 + (4/1575\,\pi)\, R(3R^4u + 10R^2u^3 + 3u^5) \\ & \times (p^{1/2})^7 + (1/36864)[\,|R - u|^7 - (R + u)^7]\,(p^{1/2})^8 + \ldots \end{aligned} \qquad \ldots(5.6)$$

in (5.2) and equating powers of $p^{1/2}$; one solves for H.

The transformed torque for particular types of rotations are

(i) $\Omega(T) = 0.5\, H(T)$

$$\begin{aligned} p\bar{M}_\infty(p) = {} & (16/3) + (16/15)(p^{1/2})^2 - (64/27\,\pi)(p^{1/2})^3 \\ & + (176/315)(p^{1/2})^4 - (896/675\,\pi)(p^{1/2})^5 + ((608/2835) \\ & + (256/243\,\pi^2))(p^{1/2})^6 - (42304/55125)(p^{1/2})^7 + \ldots \end{aligned} \qquad \ldots(5.7)$$

(ii) $\Omega(T) = 0.5 \exp(-T) H(T)$

$$p\bar{M}_\infty(p) = (16/3)(p^{1/2})^2 - (64/15)(p^{1/2})^4 - (64/27\pi)(p^{1/2})^5 + (304/63)(p^{1/2})^6 + (704/675\pi)(p^{1/2})^7 + ((256/243\pi^2) - (1775464/7257760))(p^{1/2})^8 - (606568/330750)(p^{1/2})^9 + \ldots \quad \ldots(5.8)$$

(iii) $\Omega(T) = 0.5\, TH(T)\, H(1-T)$

$$p\bar{M}_\infty(p) = (8/3)(p^{1/2})^2 - (23/18)(p^{1/2})^4 - (32/27\pi)(p^{1/2})^5 + (62/105)(p^{1/2})^6 + (256/2025\pi)(p^{1/2})^7 + ((128/243\pi^2) - (29867/241920))(p^{1/2})^8 - (6008/99225) \times (p^{1/2})^9 + \ldots. \quad \ldots(5.9)$$

Application of Schapery's algorithm ($p = 0.5/T$) gives the asymptotic torque for large time.

Semi-infinite Medium

When the disks rotate with equal and opposite angular velocities, the velocity field vanishes on the plane $Z = h/2$, which acts as a rigid wall. Since $\bar{\Omega}_0(p) + \bar{\Omega}_h(p) = 0$ (2.12) gives $H_1(R, p) = 0$ and writing

$$H_2(R, p) = (8/\pi^{1/2})\, f(R, p) \quad \ldots(5.10)$$

the velocity field is obtained as

$$\bar{V}(R, Z, p) = (4/\pi)\int_0^1 f(\sigma, p)\, d\sigma \times \int_0^\infty \frac{\xi}{(\xi^2+p)^{1/2}}\left[\begin{array}{l}\exp[-(\xi^2+p)^{1/2}\,|Z|\\ -\exp[-(\xi^2+p)^{1/2}|Z-h|]\end{array}\right] \times J_1(\xi R)\sin(\sigma\xi)\, d\xi \quad \ldots(5.11)$$

where $f(R, p)$ is the solution of the integral equation

$$f(R, p) = \bar{\Omega}(p)\, R - \int_0^1 [E_1(R, u, p) - E_2(R, u, p)]\, f(u, p)\, du. \quad \ldots(5.12)$$

The torque on the disks is

$$\bar{M}(p) = 32\int_0^1 \sigma f(\sigma, p)\, d\sigma. \quad \ldots(5.13)$$

If, in particular, the disks are rotating with uniform angular velocities $\Omega_0 = -\Omega_h = 0.5$, one gets

$$V_\infty(R, Z, \infty) = (2/\pi) \int_0^1 f_3(\sigma) \int_0^\infty [\exp(-|Z|\xi) - \exp(-|Z-h|\xi)] \times J_1(\xi R) \sin(\sigma\xi)\, d\xi\, d\sigma \quad \ldots(5.14)$$

where $f_3(\sigma)$ is a solution of

$$f_3(R) - (1/\pi) \int_0^1 \left[\frac{h}{h^2 + (R-u)^2} - \frac{h}{h^2 + (R+u)^2}\right] f_3(u)\, du = R \quad \ldots(5.15)$$

(5.14) and (5.15) agree with Cooke (1956).

The magnitude of the torque on either disk is

$$M_\infty(\infty) = (16/3) + (64/9\pi)(1/h^3) - (256/15\pi)(1/h^5) + (256/27\pi^2)(1/h^6) + (1536/35)(1/h^7) - (512/45\pi^2)(1/h^8) + \ldots \quad \ldots(5.16)$$

6. Discussion

Figure 1 gives the ratio of unsteady torque to steady-state torque when

$$(i)\ \Omega_0(T) = 0.5\, H(T),\ \Omega_h(T) = 0 \text{ and } (ii)\ \Omega_0(T) = -\Omega_h(T) = 0.5\, H(T)$$

for various h. It is supported by Table I which indicates that the torques on the rotating and stationary disk counteract, since the disturbance due to the rotating disk exerts

TABLE I

Unsteady torque

$T \backslash h$	$\Omega_0(T) = 0.5\, H(T),\ \Omega_h(T) = 0$						$\Omega_0(T) = -\Omega_h(T) = 0.5\, H(T)$		
	0.1		1.0		5.0		0.1	1.0	5.0
	$M_0(T)$	$-M_h(T)$	$M_0(T)$	$-M_h(T)$	$M_0(T)$	$-M_h(T)$	$M_0(T)$	$M_0(T)$	$M_0(T)$
1.0	13.0147	9.6277	5.7507	0.5670	5.6927	0.0023	22.6424	6.3177	5.6950
5.0	12.8700	9.6658	5.5010	0.6530	5.4210	0.0088	22.5358	6.1539	5.4298
50.0	12.8298	9.6777	5.4317	0.6827	5.3434	0.0150	22.5075	6.1143	5.3584
*10000.0	12.8251	9.6797	5.4229	0.6870	5.3334	0.0165	22.5048	6.1099	5.3499

*Steady-state reached.

a coupling force on the static disk in its own sense—a fact confirmed by (4.2). The unsteady torque on the rotating disk remains always greater than the steady-state torque whereas on the static disk the behaviour is reversed. Curves 4 and 5 show that the presence of a static disk at a distance $h \geqslant 5$ hardly has any effect on the rotating disk.

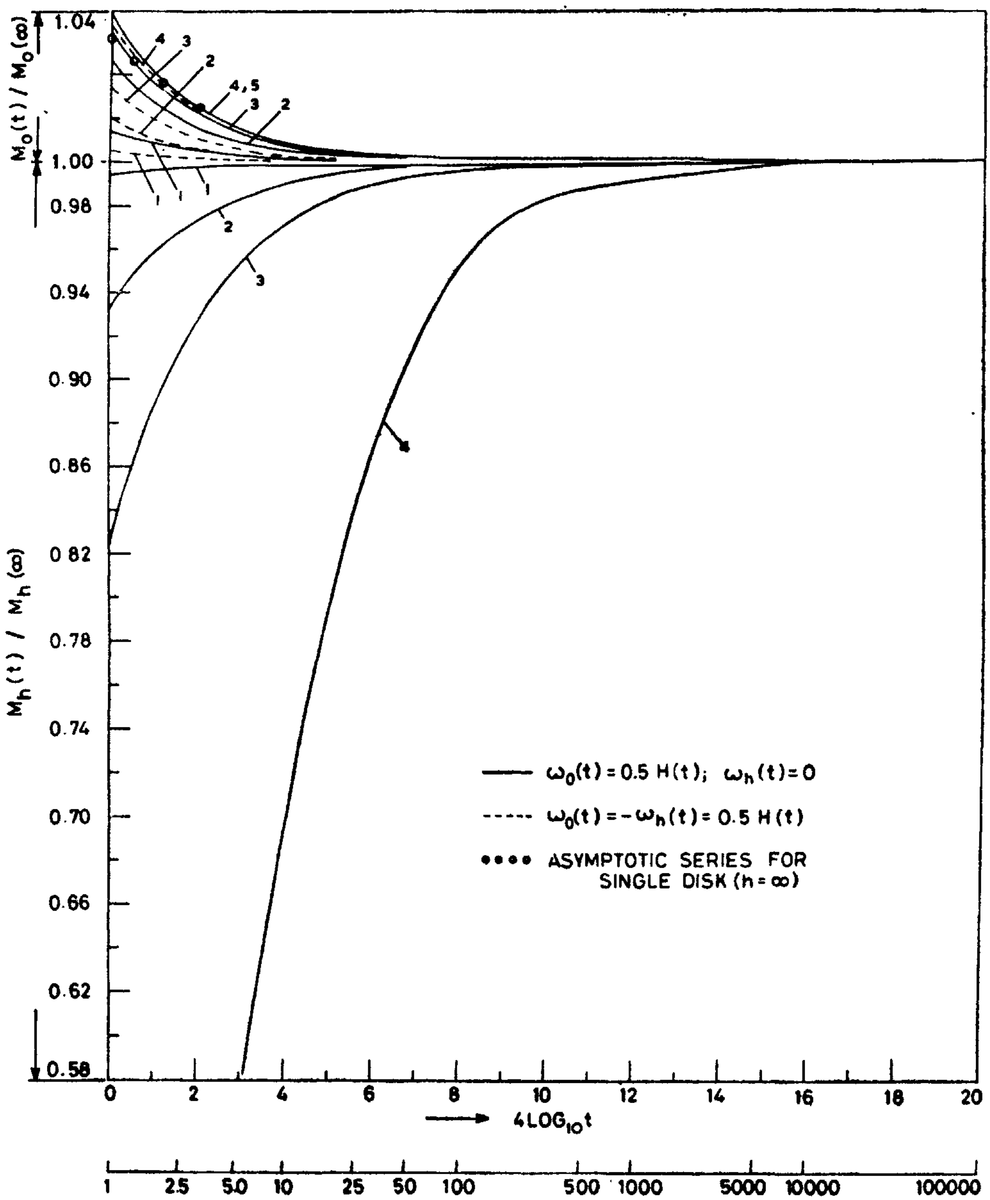

FIG. 1. Ratio of unsteady torque to steady-state torque when $\omega_0(t) = 0.5\,H(t)$, for different h : (1) 0.1, (2) 0.5, (3) 1.0, (4) 5.0, (5) ∞.

A natural question arises whether one can bound h from below, for which the presence of a static disk does not influence the rotating disk at a given time T. When $\bar{\Omega}_h(p) = 0$ and $|E_2| \ll 1$; one gets from (2.12), (2.21) and (5.3) $M_0(T) \approx M_\infty(T)$. Laplace inversion of E_2 is

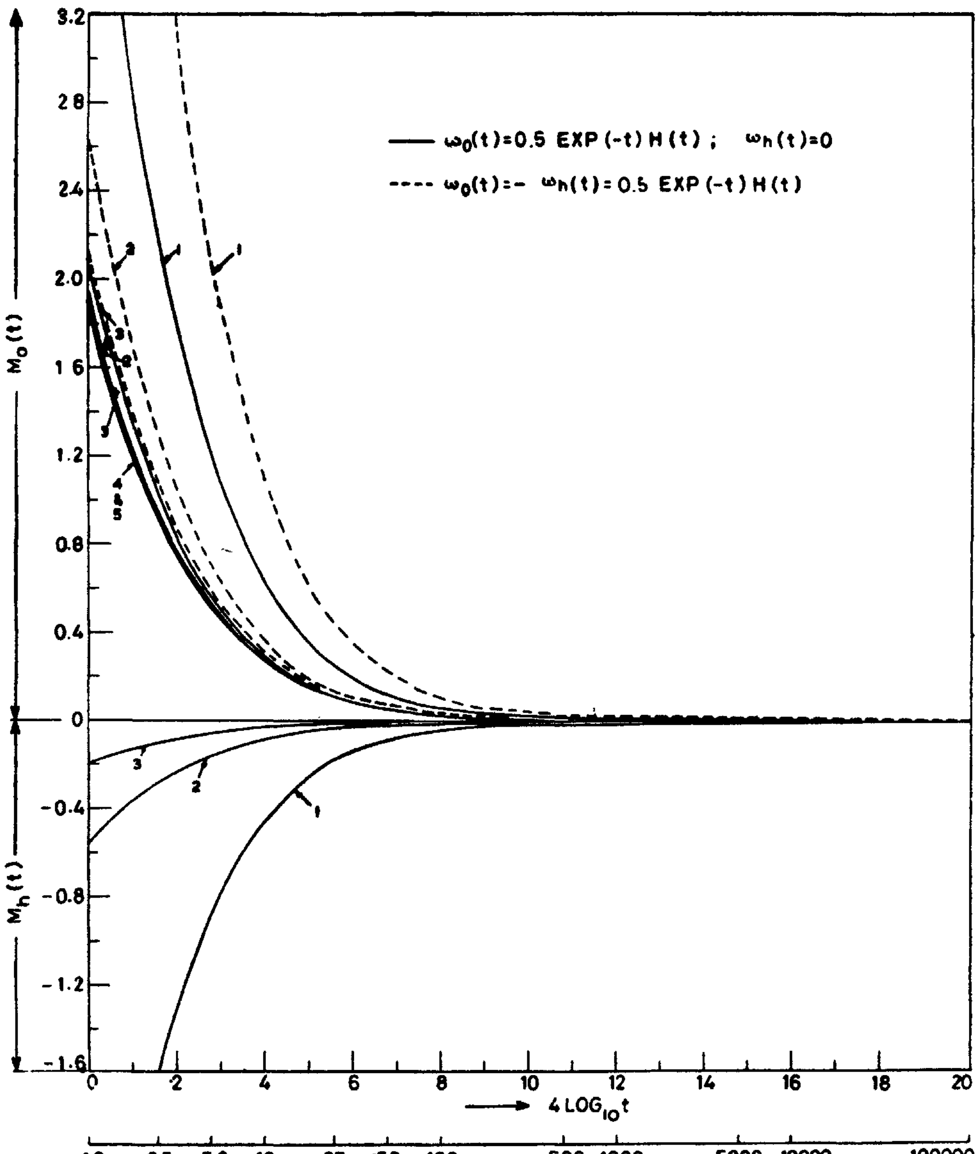

FIG. 2. Unsteady torque when $\omega_0(t) = 0.5 \exp(-t) H(t)$ for different h : (1) 0.1, (2) 0.5, (3) 1.0, (4) 5.0, (5) ∞.

$$L^{-1}[E_2(R, u, p)] = (1/(2\pi)^{3/2}) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{(2n)!} \times [(u+R)^{2n} - (u-R)^{2n}] \frac{\exp(-h^2/4T)}{T^{(2n+3)/2}} \quad \ldots(6.1)$$

an absolutely convergent alternating series with monotonically decreasing terms.

Since $0 < u, R < 1$

$$L^{-1}[E_2(R, u, p)] < \frac{1}{(\pi T)^{5/2}} \exp(-h^2/4T). \quad \ldots(6.2)$$

Therefore, given a proper degree of precision 10^{-N} and specified time T, one can calculate h from the inequality

$$h^2 > 4T \log_e\left(\frac{10^N}{\pi^{3/2}\, T^{5/2}}\right) \quad \ldots(6.3)$$

provided $\quad 10^N > \pi^{3/2}\, T^{5/2}$.

Figure 2 gives the results for unsteady torque when $\Omega_0(T) = 0.5 \exp(-T)\, H(T)$ and $\Omega_h(T) = 0$ or $-\Omega_0(T)$ for various h. From Fig. 3 it is observed that the shearing torque on the upper disk increases in magnitude initially and after reaching a peak value decays to zero with time. As h increases, $|M_h(T)|$ decreases but the time taken to record its maximum value increases.

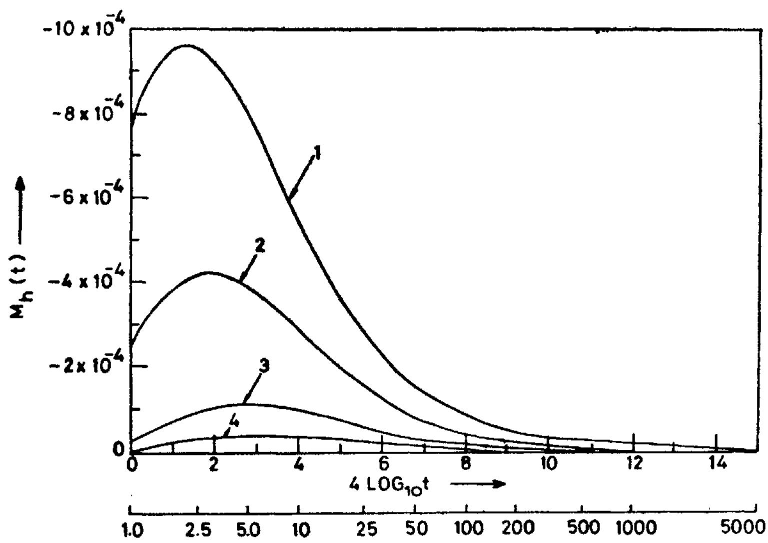

FIG. 3. Unsteady torque on the upper disk when $\omega_0(t) = 0.5 \exp(-t)\, H(t)$; $\omega_h(t) = 0$ for $h \geqslant 5$: (1) 5, (2) 6, (3) 8, (4) 10.

Figure 4 gives the steady-state torque on either disk and is supplemented by Table II. They indicate that for a given Ω_0 and a fixed h, as Ω_h increases from zero to Ω_0 and beyond, the torque on the upper disk increases from negative to positive value, crossing the zero level at some stage when the shear-stress distribution is continuous on the disk-plane ($Z = h$) for all R, in spite of its presence in the medium.

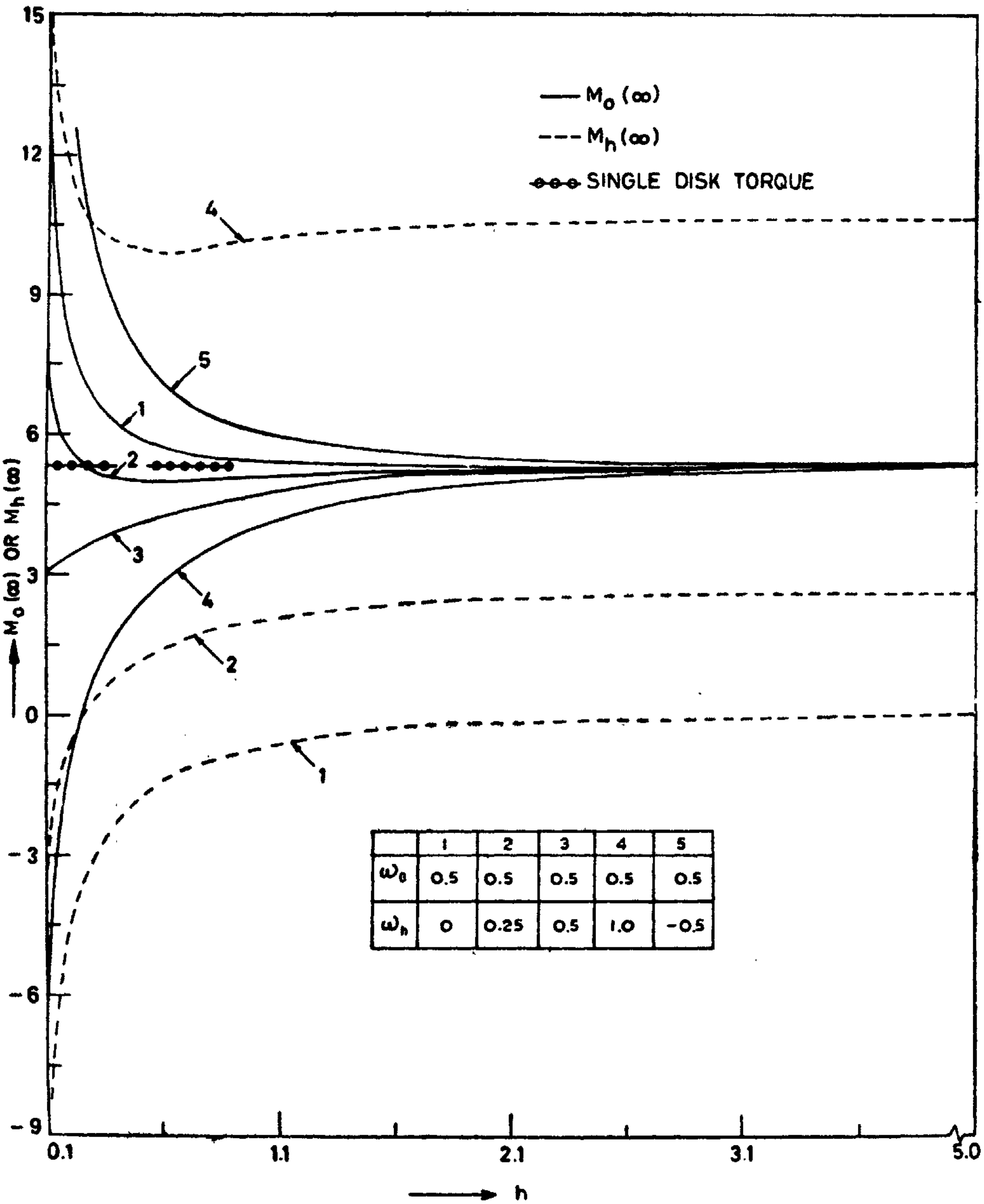

FIG. 4. Steady-state torque for different kinds of rotations as h varies

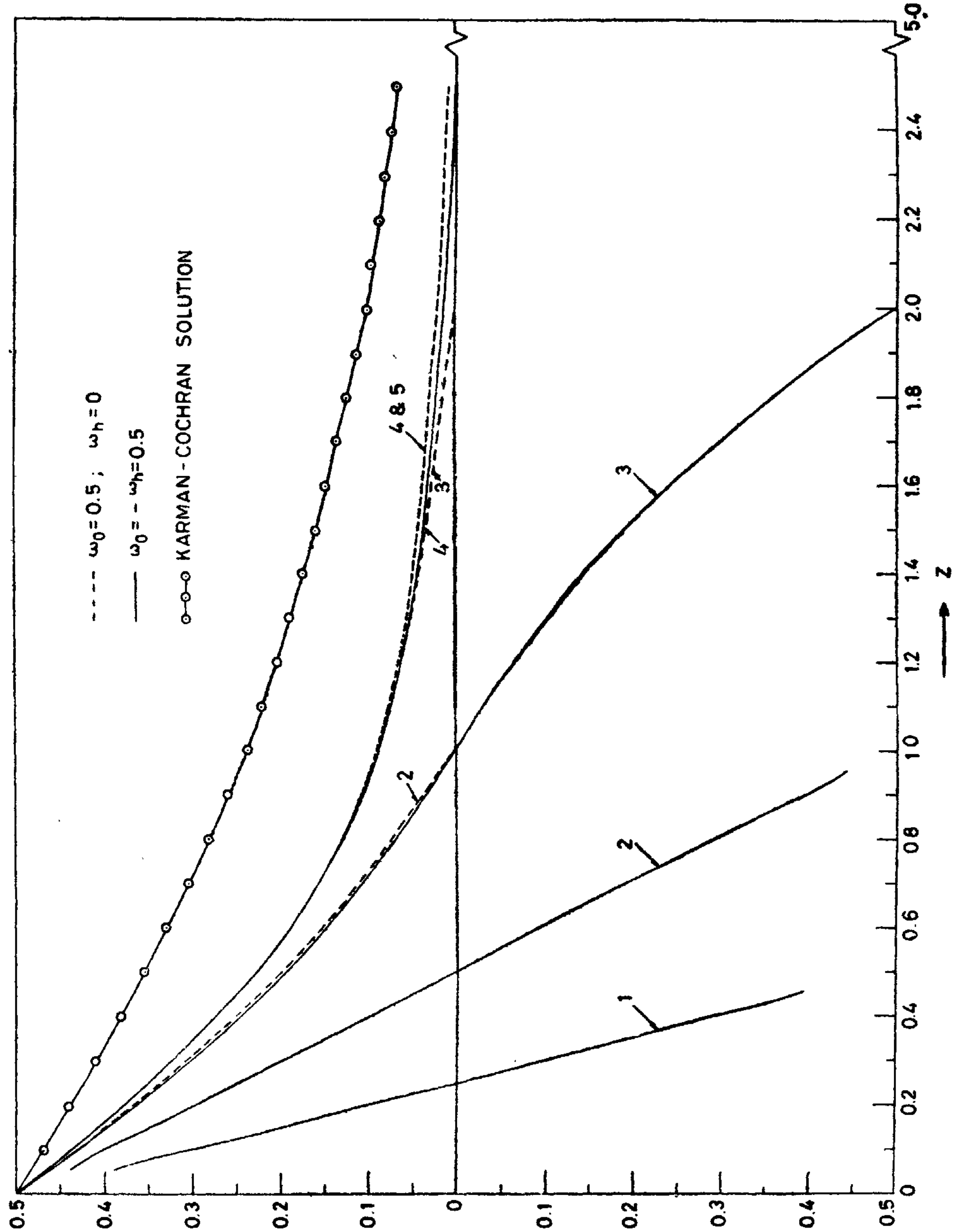

FIG. 5. Swirl between the disks along the axis of symmetry in the case of steady-state motion for different h : (1) 0.5, (2) 1.0, (3) 2.0, (4) 5.0, (5) ∞.

TABLE II

Steady-state torque

$\Omega_0 = 0.5$ / $\Omega_h \to$	0.0		0.1		0.25		0.5	1.0		−0.5
h	$M_0(\infty)$	$M_h(\infty)$	$M_0(\infty)$	$M_h(\infty)$	$M_0(\infty)$	$M_h(\infty)$	$M_0(\infty) = M_h(\infty)$	$M_0(\infty)$	$M_h(\infty)$	$M_0(\infty) = -M_h(\infty)$
0.1	12.8505	−9.7344	10.9036	−7.1643	7.9833	−3.3091	3.1161	−6.6183	15.9665	22.5848
0.4	6.2069	−2.2747	5.7520	−1.0333	5.0696	0.8288	3.9322	1.6576	10.1391	8.4815
0.6	5.7029	−1.4199	5.4189	−0.2793	4.9930*	1.4316	4.2830	2.8631	9.9859*	7.1228
1.0	5.4229	−0.6870	5.2855*	0.3976	5.0794	2.0245	4.7359	4.0489	10.1588	6.1099
5.0	5.3334	−0.0165	5.3301	1.0501	5.3251	2.6502	5.3169	5.3030	10.6502	5.3499
100.0	5.3333	0.0000	5.3333	1.0667	5.3333	2.6667	5.3333	5.3333	10.6666	5.3333

*Minimum torque.

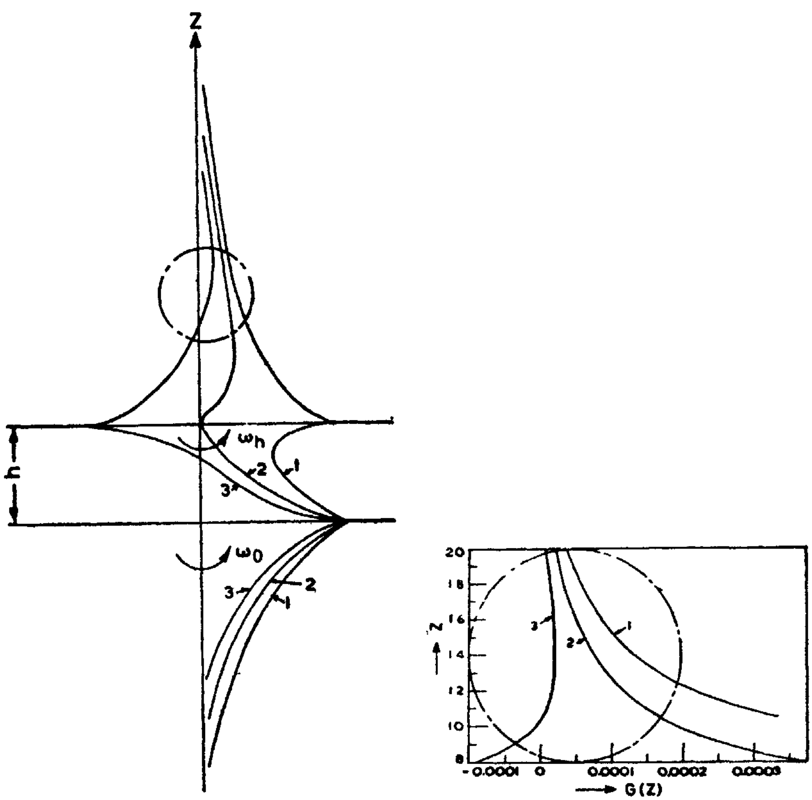

FIG. 6. Typical steady-state swirl in three distinct cases : (1) $\omega_0 > \omega_h > 0$, (2) $\omega_0 > \omega_h = 0$, (3) $\omega_h < 0$; $\omega_0 > | \omega_h |$

If $0 < \Omega_h < \Omega_0$, for small h the upper disk acts like a retarder, as a result the lower disk experiences more torque than the steady-state torque on a single disk but less than the correrponding torque when $\Omega_h = 0$. The retarding effect gradually decreases with increasing h, thereby the torque on the lower disk keeps on reducing, crosses the level of single disk torque and reaches a minimum. From that stage onwards, the upper disk acts like an accelerator, helping the torque on the lower disk to increase till it settles to the single disk torque from below, for sufficiently large h. This drag reduction phenomenon (particularly significant to hydraulic machine efficiencies) can be observed from curves 2 and 4 and from Table II at the asterisk stages.

The strength of the swirl velocity along the axis of rotation is calculated in order to see the influence of the second disk on the flow pattern and projected in Fig. 5. Comparing with G-profiles of von Karman-Schlichting (1962) one notes that the swirl is faster and dies lately with Z than in the present case $h = \infty$ and agree reasonably near and far-off the disk, with the maximum deviation 0.15 occurring in the neighbourhood of $Z = 1$. However, a similar comparison with the results of Bentwich (1969) shows a perfect agreement.

The behaviour of the swirl beyond the upper disk is projected schematically in Fig. 6. As Ω_h goes from positive to zero value, the swirl changes its direction of concavity and goes to zero gradually with Z, giving a bulge. Further when Ω_h changes sign the bulge shifts further up as the swirl crosses the Z axis and disappears when $\Omega_0 = -\Omega_h$.

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