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DEFLECTION OF A CLAMPED CURVILINEAR PLATE UNDER AN ISOLATED FORCE AT ARBITRARY POINT

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Some rational mapping functions are used to derive exact solutions in closed form for the deflections of certain thin elastic isotropic finite curvilinear clamped plates normally loaded by isolated forces at arbitrary points. Numerical evaluation of deflections for selected shapes of the plates are presented.

1. INTRODUCTION

It is known (Bassali 1959) that for a finite thin elastic plate bounded by a clamped closed contour Γ and normally loaded by an isolated normal force P at an arbitrary point $Q(z_0)$ inside Γ, the deflection $w(x, y)$ measured positively upwards is given by

$$w = 2k \, [\, | z - z_0 |^2 \ln | (1 - \bar{\zeta}_0\zeta)/(\zeta - \zeta_0) | + \text{Re}\, \{\bar{z}F\{\zeta) + f(\zeta)\}], \quad \ldots(1)$$

$$k = P/(16\pi D).$$

Here

$z = x + iy = z(\zeta)$, $\zeta = \xi + i\eta = \rho e^{i\theta}$, $z'(\zeta) \neq 0, \infty$ for $\rho < 1$ maps the region inside Γ conformally on the area inside the unit circle γ, ζ_0 corresponds to z_0, D is the flexural rigidity and $F(\zeta)$, $f(\zeta)$ are regular functions of $\zeta(\, | \zeta | \leqslant 1)$ to be determined from the boundary conditions.

$$w = 0, \frac{\partial w}{\partial \zeta} = 0 \quad \text{on } \gamma. \quad \ldots(2)$$

These conditions can be taken in the forms

$$\overline{z(\sigma)}\, F(\sigma) + z(\sigma)\, \overline{F(\sigma)} + f(\sigma) + \overline{f(\sigma)} = 0 \quad \ldots(3)$$

$$\sigma\, [z'(\sigma)\, \overline{F(\sigma)} + \overline{z(\sigma)}\, F'(\sigma) + f'(\sigma)] = t(\rho_0) \mid (z(\sigma) - z_0(\zeta_0))/(\sigma - \zeta_0) \mid^2 \quad \ldots(4)$$

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where $t(\rho) = 1 - \rho^2$ and $\sigma = e^{i\theta}$ is any point on γ.

Closed expressions were obtained (Bassali and El-Sirafy 1976) for the deflections of clamped and singularly loaded curvilinear finite plates by using the transformations

$$z = c\zeta(1 + \alpha\zeta^{\nu})/(1 + \beta\zeta^{\mu}), \quad c > 0, \quad (\nu, \mu = 1, 2) \qquad \ldots(5)$$

where α and β are real constants.

In the present work, the plates taken in the z-plane, can be mapped on the region inside the unit circle in the ζ-plane by one of the following rational functions

$$z = c\zeta(1 + m\zeta^{\nu})/(1 + n\zeta^{3}), \quad (\nu = 1, 2) \qquad \ldots(6)$$

$$z = c\zeta(1 + p\zeta)/(1 + q\zeta + h\zeta^{2}) \qquad \ldots(7)$$

where $c > 0$, m, n, p, q, h are real parameters and $z'(\zeta) \neq 0, \infty$ inside γ.

Several previously known solutions appear as special cases of the deflections given here.

2. Method of Solution

The tentative method of solution (Bassali 1959) adopted here is to assume suitable forms for the potentials $F(\zeta)$ and $f(\zeta)$ and we then show that they fit the conditions (3) and (4).

(i) *First Mapping Function*

We now consider plates mapped on $|\zeta| \leqslant 1$ by the conformal transformation (6). The parametric equations of Γ are

$$\left.\begin{aligned} \frac{x}{c} &= \frac{\cos\theta + n\cos 2\theta + m\cos(1+\nu)\theta + mn\cos(2-\nu)\theta}{1 + n^2 + 2n\cos 3\theta} \\ \frac{y}{c} &= \frac{\sin\theta - n\sin 2\theta + m\sin(1+\nu)\theta - mn\sin(2-\nu)\theta}{1 + n^2 + 2n\cos 3\theta} \end{aligned}\right\} \qquad \ldots(8)$$

The two functions $F(\zeta)$ and $f(\zeta)$ appearing in (1) will be assumed as

$$\left.\begin{aligned} F(\zeta) &= c\sum_{\nu=0}^{3} A_{\nu}\zeta^{\nu}/(1 + n\zeta^{3}) \\ f(\zeta) &= c^2\sum_{\nu=0}^{3} B_{\nu}\zeta^{\nu}/(1 + n\zeta^{3}) \end{aligned}\right\} \qquad \ldots(9)$$

in which A_0, A_2, A_3 B_2 and B_3, are in general, complex constants, A_1 and B_0 may be taken as real and B_1 may be taken as zero. The values of these constants will be determined so as to satisfy the boundary conditions (3) and (4).

Case (*a*) : $\nu = 1$ — Here we have the transformation

$$z = c\zeta(1 + m\zeta)/(1 + n\zeta^3). \qquad \ldots(10)$$

Substituting from (9), (10) in (3), (4) and equating coefficients of powers of σ to zero we obtain the equations

$$2A_1 + m(A_2 + \bar{A}_2) + 2B_0 + n(\bar{B}_3 + B_3) = 0,$$

$$A_2 + mA_3 + \bar{A}_0 + mA_1 + n\bar{B}_2 = 0,$$

$$A_3 + m\bar{A}_0 + B_2 = 0, \;\; B_3 + nB_0 = 0,$$

$$-mn\bar{A}_0 - nB_2 = nC_2,$$

$$-2n\bar{A}_0 - mnA_1 - nA_2 = nC_1,$$

$$-4nA_1 - mn(A_2 + \bar{A}_2) + 3(B_3 - nB_0) = nC_0,$$

$$2m\bar{A}_0 - 2n\bar{A}_2 - mn\bar{A}_3 + 3(A_3 - nA_0) - 2mnA_1 + (2 - n^2)\,B_2 = C_2 + n\bar{C}_1,$$

$$\bar{A}_0 + 2mA_1 - 2n\bar{A}_3 + 2A_2 + 3m(A_3 - nA_0) = C_1 + n\bar{C}_2,$$

$$2A_1 + 2m(A_2 + \bar{A}_2) + 3n(B_3 - nB_0) = C_0,$$

$$\bar{A}_2 + 2m\bar{A}_3 + mA_1 + 2nB_2 = \bar{C}_1, \;\; \bar{A}_3 = \bar{C}_2,$$

where

$$C_0 = t(\rho_0)[(1 + n^2\rho_0^2)(1 + m^2\rho_0^2 + 2m\xi_0) + m^2 + n^2\rho_0^4 - mn(\zeta_0^2 + \bar{\zeta}_0^2)]\,|\,1 + n\zeta_0^3\,|^{-2},$$

$$C_1 = t(\rho_0)\,[m + mn^2\rho_0^4 - mn(1 + \rho_0^2)\,\zeta_0 + (m^2 + n^2\rho_0^2)\,\bar{\zeta}_0 - n(1 + m^2)\,\zeta_0^2]\,|\,1 + n\zeta_0^3\,|^{-2},$$

$$C_2 = -\,n\zeta_0 t(\rho_0)\,(1 + m^2\rho_0^2 + 2m\xi_0)/\,|\,1 + n\zeta_0^3\,|^{2}.$$

It is found that these equations are consistent in the unknowns for which the following values are obtained:

$$A_0 = (1 - m^2n^2)^{-1}\,[mnC_1 - \bar{C}_1 - n(1 + m^2)\,C_2 + m(1 + n^2)\,\bar{C}_2],$$

$$A_1 = \tfrac{1}{2}J\,[(1 - n^2)\,C_0 - m(2 + n^2)\,\{C_1 + \bar{C}_1 + 2(n - m)\,(C_2 + \bar{C}_2) + 2mn(A_0 + \bar{A}_0)\}],$$

$$A_2 = C_1 - m(A_1 - 2nA_0) - 2mC_2 + 2n\bar{C}_2, \;\; A_3 = C_2,$$

$$B_0 = -\,(\tfrac{1}{2}C_0 + A_1)/(2 + n^2), \;\; B_2 = -\,C_2 - m\bar{A}_0, \;\; B_3 = -nB_0,$$

where $\quad J = (1 - 2m^2 + 2n^2 - m^2n^2)^{-1}.$

When these values are introduced in (9) and the resulting functions inserted in (1) it is found that the deflection at $z(\zeta)$ due to the isolated force P at $z_0(\zeta_0)$ is given by

$$\frac{w}{2kc^2} = \left|\frac{z-z_0}{c}\right|^2 \ln\left|\frac{1-\bar{\zeta}_0\zeta}{\zeta-\zeta_0}\right| + \frac{t(\rho_0)\,t(\rho)}{|1+n\zeta_0^3|^2\,|1+n\zeta^3|^2}$$

$$\times \operatorname{Re}\{T(\rho_0, \rho) + H(\rho_0, \rho)\,\bar{\zeta}_0\zeta + m\,[E(\rho_0, \rho)\,\zeta_0 + E(\rho, \rho_0)\,\zeta]$$

$$+ mn(1 + m\zeta_0\zeta)\,(\rho_0^2\zeta^2 + \rho^2\zeta_0^2)$$

$$+ n\zeta_0\zeta\,[\zeta_0 + \zeta + m(\zeta_0\zeta + \rho_0^2 + \rho^2 + \beta t(\rho_0)\,t(\rho))]\} \qquad \ldots(11)$$

where

$$T(\rho_0, \rho) = \tfrac{1}{2}J\,[(m^4 - 2m^2 - 1)\,(1 + n^4\rho_0^4)\,(1 + n^4\rho^4)$$

$$+ \frac{4m^2 t(\rho_0)\,t(\rho)}{1+mn}\,(1 - n^2\rho_0^2)\,(1 - n^2\rho^2)$$

$$+ (m^4 - n^4 - 4m^2n^2 - 4m^2)\,\rho_0^2\rho^2 + (m^4 + m^2n^2$$

$$+ 3m^2 - n^2)\,(\rho_0^2 + \rho^2)\,(1 + n^2\rho_0^2\rho^2)],$$

$$H(\rho_0, \rho) = H(\rho, \rho_0) = (m^2 - n^2)\,\rho_0^2\rho^2 - m^2(\rho_0^2 + \rho^2) - \beta t(\rho_0)\,t(\rho),$$

$$\beta = (m^2 - n^2)/(1 - m^2n^2),$$

$$E(\rho, \rho_0) = (n^2\rho_0^2 - 1)\left(\frac{t(\rho)\,t(\rho_0)}{1+mn} - \rho^2\rho_0^2\right) - (\rho^2 + \rho_0^2).$$

Case (*b*) — The value $\nu = 2$ yields the function

$$z = c\zeta(1 + m\zeta^2)/(1 + n\zeta^3). \qquad \ldots(12)$$

Following the usual procedure of determining the constants involved in (9) so as to satisfy the two boundary conditions (3), (4) it is found, after extensive algebraic calculations, that the vertical displacement w is furnished by

$$\frac{w}{2kc^2} = \left|\frac{z-z_0}{c}\right|^2 \ln\left|\frac{1-\bar{\zeta}_0\zeta}{\zeta-\zeta_0}\right| + \frac{t(\rho_0)\,t(\rho)}{|1+n\zeta_0^3|^2\,|1+n\zeta^3|^2}$$

$$\times \operatorname{Re}\{T(\rho_0, \rho) - m(\zeta_0^2 + \zeta^2) + [n(\zeta_0 + \zeta) - mL(\rho_0, \rho)]\,\zeta_0\zeta$$

$$+ \frac{mn}{1+2m}\,[H(\rho_0, \rho)\,\zeta + H(\rho, \rho_0)\,\zeta_0]$$

$$+ [E(\rho_0, \rho) + mn(\rho_0^2\zeta + \rho^2\bar{\zeta}_0) - m^2\bar{\zeta}_0\zeta]\,\zeta\bar{\zeta}_0\} \qquad \ldots(13)$$

where

$$2T(\rho_0, \rho) = J\,[(1 + 2m)\,(m^4 - n^4) + 2m^2n^2(1 + m)]\,(\rho_0^4 + \rho_0^2 - 2)$$

$$\times (\rho^4 + \rho^2 - 2) - (m^2 + n^2)(\rho_0^4 + \rho_0^2 + \rho^4 + \rho^2) + 2n^2 + m^2 - 1,$$

$$J = (1 + 2m)^{-1} (1 - 3m^2 + 2n^2)^{-1},$$

$$L(\rho_0, \rho) = 1 + t(\rho_0)\, t(\rho)\, (m^2 + 2n^2)/(1 - 4m^2),$$

$$H(\rho_0, \rho) = \rho_0^2(\rho_0^2 + 1)(1 + m + m\rho^2) + \rho^2 - 1,$$

$$E(\rho_0, \rho) = (1 - 4m^2)^{-1} [2m^4 + n^2 + (\rho_0^2 + \rho^2)(2m^4 - m^2 - n^2) + 2m^2(2n^2 + m^2)\rho_0^2 \rho^2].$$

Figure 1 shows the boundary of the mapping function (12) for the case $m = n = 0.1$. The deflection of the plate obtained by (13) is also presented in the same figure for isolated loads along the axis of symmetry.

(ii) *Second Mapping Function*

Finally we consider the transformation (7). In this case the parametric equations of Γ are

$$\left.\begin{aligned} \frac{x}{c} &= [(q + ph) + (1 + h + pq)\cos\theta + p\cos 2\theta]/R(\theta), \\ \frac{y}{c} &= [(1 - h + pq)\sin\theta + p\sin 2\theta]/R(\theta), \\ R(\theta) &= 1 + q^2 + h^2 + 2q(1 + h)\cos\theta + 2h\cos 2\theta. \end{aligned}\right\} \quad \ldots(14)$$

Here we assume that

$$F(\zeta) = c\,\frac{A_0 + A_1\zeta + A_2\zeta^2}{1 + q\zeta + h\zeta^2}, \quad f(\zeta) = c^2\,\frac{B_0 + B_2\zeta^2}{1 + q\zeta + h\zeta^2},$$

where A_1 and B_0 are real. When the five constants are determined it is found that the deflection is

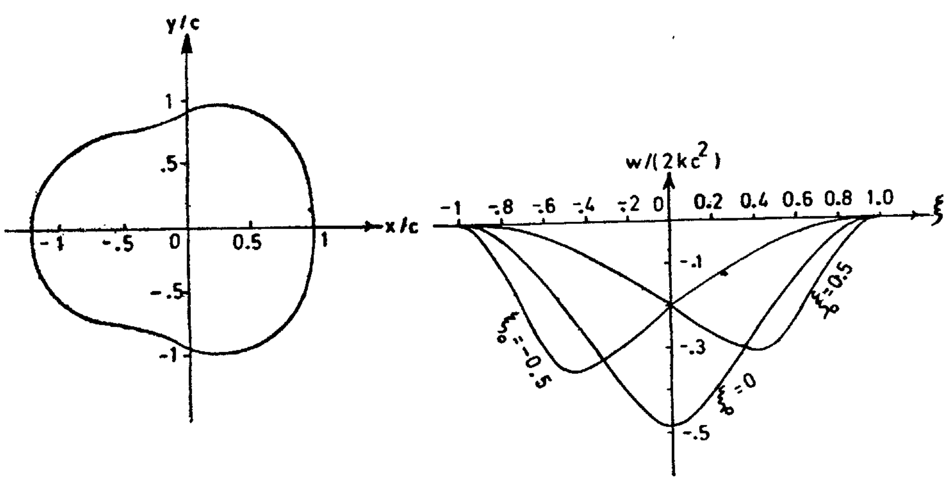

FIG. 1. $z = c\,(1 + 0.1\zeta^2)/(1 + 0.1\zeta^3)$, $\eta_0 = 0$.

$$\frac{w}{2kc^2} = \frac{t(\rho_0)\, t(\rho)}{|1 + q\zeta_0 + h\zeta_0^2|^2 \, |1 + q\zeta + h\zeta^2|^2}$$

$$\times \operatorname{Re} \{(h - pq)\, \zeta_0 \zeta - p^2 \bar{\zeta}_0 \zeta + p(h - pq)\, (\rho_0^2 \zeta + \rho^2 \zeta_0)$$

$$- p(\zeta + \zeta_0) + \tfrac{1}{2} J([p^2(p^2 + 2pq - 2h - 1)$$

$$- (1 + p^2)\, (h - pq)^2]\, (\rho_0^2 + \rho^2) + [p^4 - (h - pq)^4$$

$$+ 2p^2(h - pq)\, (1 + h - pq)]\, \rho_0^2 \rho^2 + p^4$$

$$+ 2p^2(1 + h - pq) - 1)\} + \left|\frac{z - z_0}{c}\right|^2 \ln \left|\frac{1 - \bar{\zeta}_0 \zeta}{\zeta - \zeta_0}\right|, \qquad \ldots(15)$$

where

$$J = [1 - 2p^2 + (pq - h)^2]^{-1}.$$

Figure 2 shows the boundary of the mapping function (7) and the deflection (15) for the case with $p = 0.2$, $q = 0.5$ and $h = 0.3$.

3. Special Cases

(1) For $\nu = 1$, $n = 0$ in (6) or $q = h = 0$ in (7) we get the function

$$z = c\zeta(1 + m\zeta) \qquad \ldots(16)$$

which maps the area inside the elliptic limacon

$$r = c(1 + 2m \cos \phi) \qquad \ldots(17)$$

on the area inside γ (see Fig. 3). The coordinates of the polar point with respect to the axes $0x$, $0y$ are $(-mc, 0)$. In this case the expression (11), (15) reduce to that given by eqn. (4.7) of Bassali and El-Sirafy (1976).

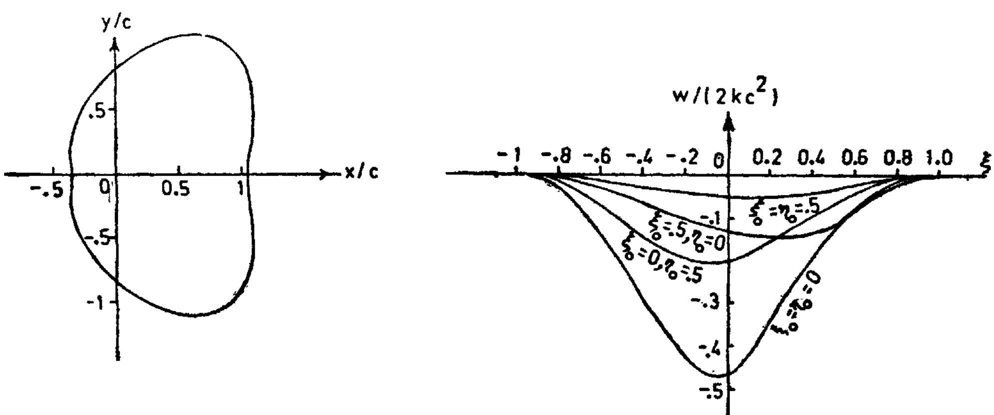

Fig. 2. $z = c\,(1 + 0.2\zeta)/(1 + 0.5\zeta + 0.3\zeta^2)$.

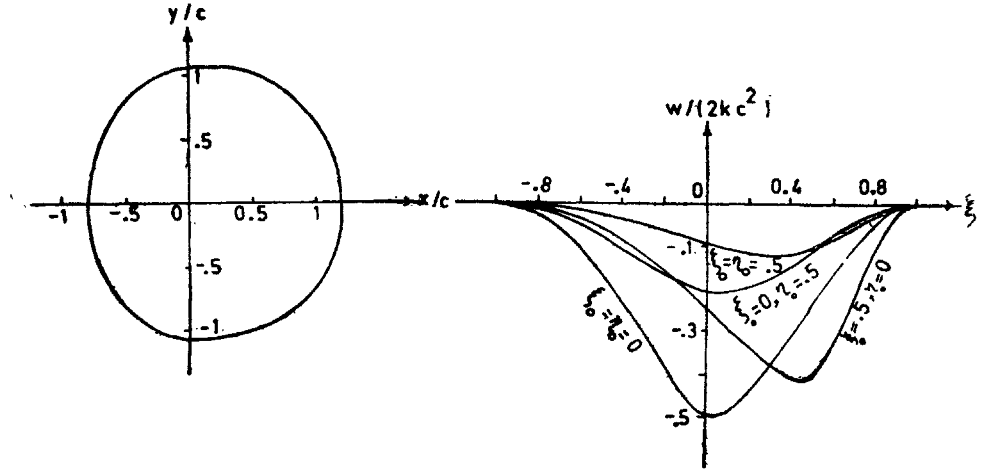

FIG. 3. $z = c\zeta(1 + 0.2\zeta)$.

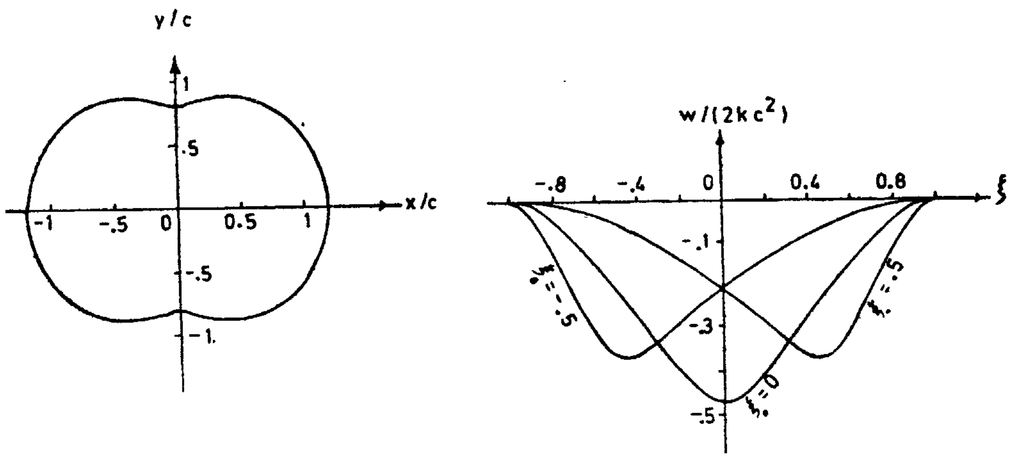

FIG. 4. $z = c\zeta(1 + 0.2\zeta^2)$, $\eta_0 = 0$.

(2) For $\nu = 2$, $n = 0$, the mapping function (6) becomes (see Fig. 4)

$$z = c\zeta(1 + m\zeta^2) \qquad \ldots(18)$$

and the plate is founded by a curvilinear polygon similar to dumb bell shape $(0 \leqslant m \leqslant \frac{1}{3})$. For $m = \frac{1}{3}$, the boundary becomes a two-cusped epicycloid.

The expression (13) reduces to that given by eqn. (4.21) of Bassali and El-Sirafy (1976).

(3) For $m = 0$ we have the mapping function

$$z = c\zeta/(1 + n\zeta^3), \quad n \leqslant \tfrac{1}{2} \qquad \ldots(19)$$

the plate is therefore bounded by three approximately circular arcs with three axes of symmetry (see Fig. 5) and the expressions (11), (13) reduce to that given by eqn. (3.88) of Bassali (1959).

(4) The formula (15) agrees with (4.15) of Bassali and El-Sirafy (1976) in the case of $q = 0$ while for $h = 0$ it simplifies to (4.33) of the same paper and the mapping functions in these cases are given by (5) with $\nu = 1$, $\mu = 1, 2$ (see Fig. 6).

(5) For $p = 0$ we get the function

$$z = c\zeta/(1 + q\zeta + h\zeta^2) \qquad \ldots(20)$$

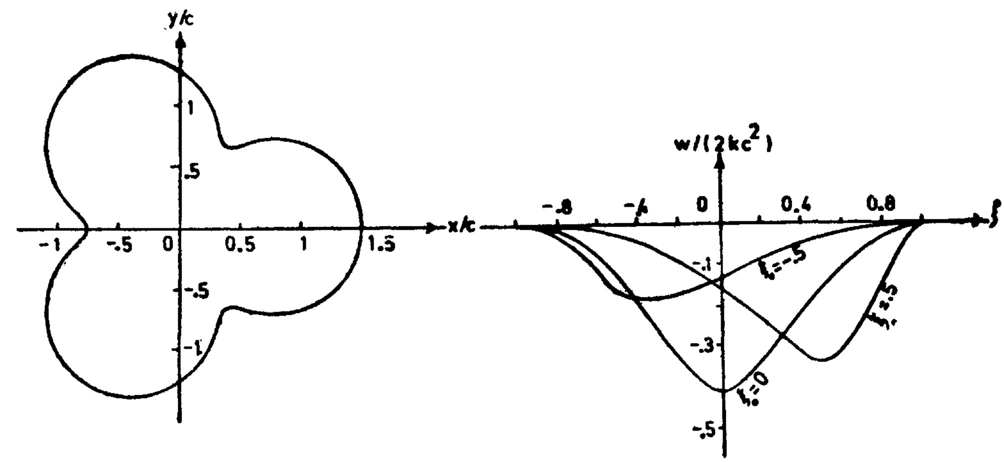

FIG. 5. $z = c\zeta/(1 - \frac{1}{3}\zeta^3)$, $\eta_0 = 0$.

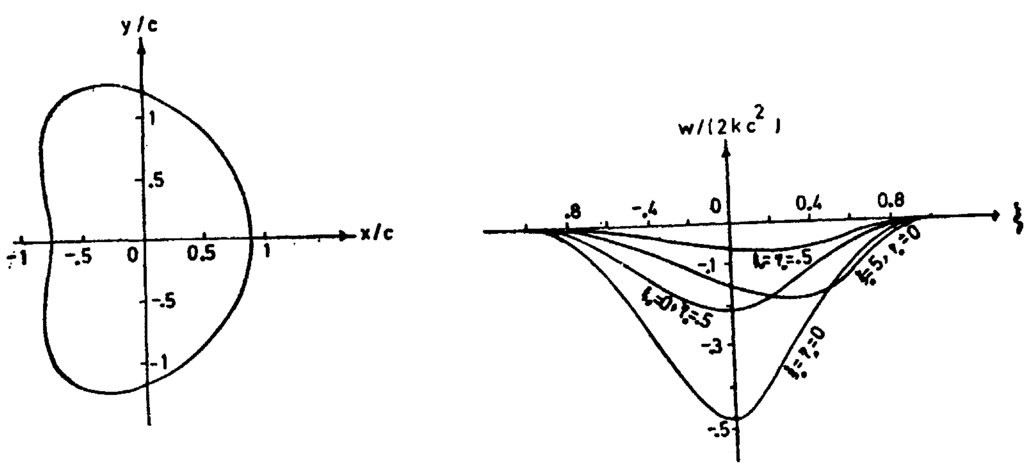

FIG. 6. $z = c\zeta(1 - 0.1\zeta)/(1 + 0.2\zeta^2)$.

and the boundary Γ in this case is the inverse of an ellipse with respect to internal point on its major axis (see Fig. 7) and when $q = 0$, the internal point coincide with the centre of the ellipse (see Fig. 8). Setting $p = 0$ in (15) yields

$$\frac{w}{2kc^2} = \left|\frac{z - z_0}{c}\right|^2 \ln\left|\frac{1 - \bar{\zeta}_0\zeta}{\zeta - \zeta_0}\right| + \frac{t(\rho_0)\, t(\rho)}{2\,|\,1 + q\zeta_0 + h\zeta_0^2\,|^2\,|\,1 + q\zeta + h\zeta^2\,|^2}$$

$$\times \left\{h(\zeta_0\zeta + \bar{\zeta}_0\bar{\zeta}) - \frac{(1 + h^2\rho_0^2)\,(1 + h^2\rho^2)}{1 + h^2}\right\} \qquad \ldots(21)$$

which agrees with eqn. (67), of Bassali (1960, p. 503).

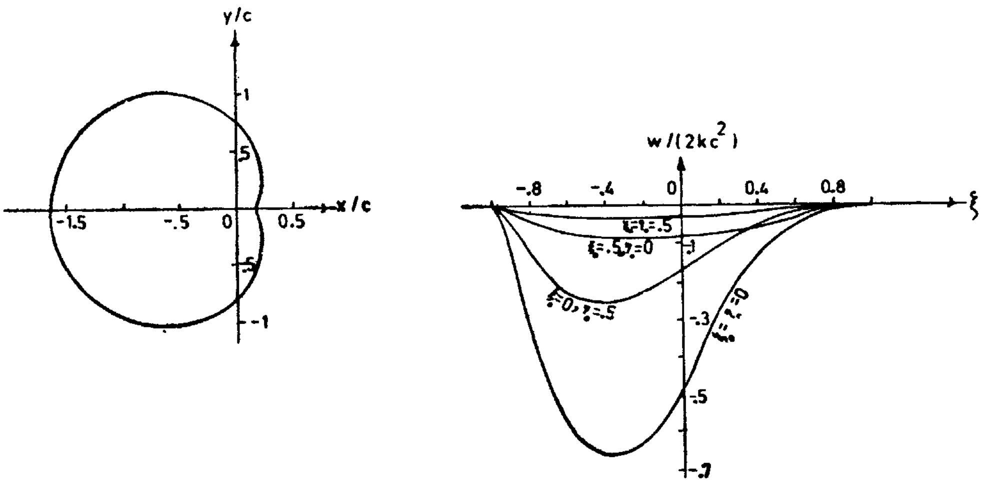

FIG. 7. $z = c\zeta/(1 + \zeta + 0.2\zeta^2)$.

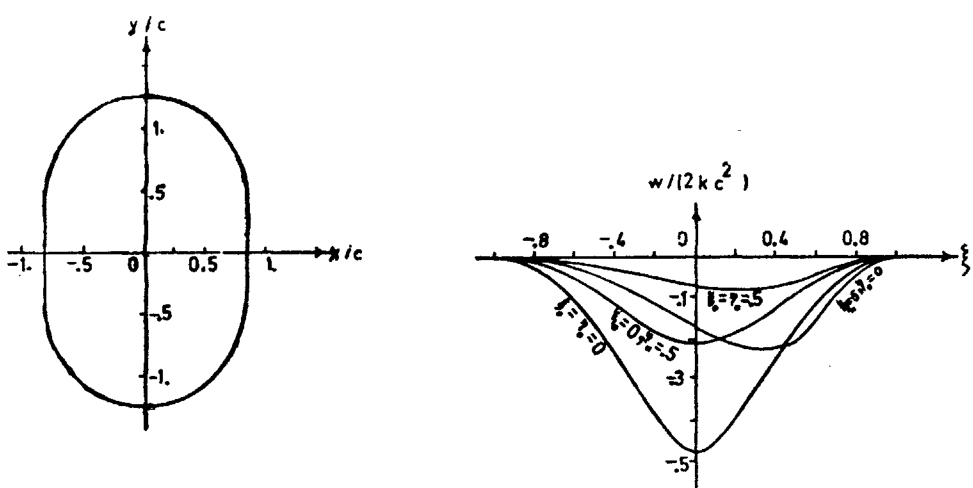

FIG. 8. $z = c\zeta/(1 + 0.2\zeta^2)$.

(6) In the special case $\nu = 1$, $n = m^3$, the mapping function (6) becomes

$$z = c\zeta/(1 - m\zeta + m^2\zeta^2)$$

and it is checked that (11) reduces to (21) with $q = -m$, $h = m^2$.

(7) Setting $\nu = 2$, $m = -p^2$, $n = -p^3$ in (6) leads to

$$z = c\zeta(1 + p\zeta)/(1 + p\zeta + p^2\zeta^2) \qquad \ldots(22)$$

and the formula (13) reduces considerably to the result obtained by putting $q = p$, $h = p^2$ in (15) (see Fig. 9).

(8) For $q = 2p$, $h = p^2$ the function (7) becomes

$$z = c\zeta/(1 + p\zeta)$$

and it can be verified that (15) simplifies to

$$\frac{w}{2kc^2} = \left|\frac{z - z_0}{c}\right|^2 \ln\left|\frac{1 - \bar{\zeta}_0\zeta}{\zeta - \zeta_0}\right| - \frac{t(\rho_0)\, t(\rho)}{2\,|\,1 + p\zeta_0\,|^2\,|\,1 + p\zeta\,|^2} \cdot \qquad \ldots(23)$$

It is easily seen that this expression is in agreement with Michell's (1901) solution for a thin clamped circular plate by a transverse force at any point.

(9) For $m = n = p = q = h = 0$, each of the transformations (6) and (7) become

$$z = c\zeta$$

and it can be verified that each of the formulae (11), (13) and (15) simplifies to Michell's solution of a circular plate $|\,z\,| = c$,

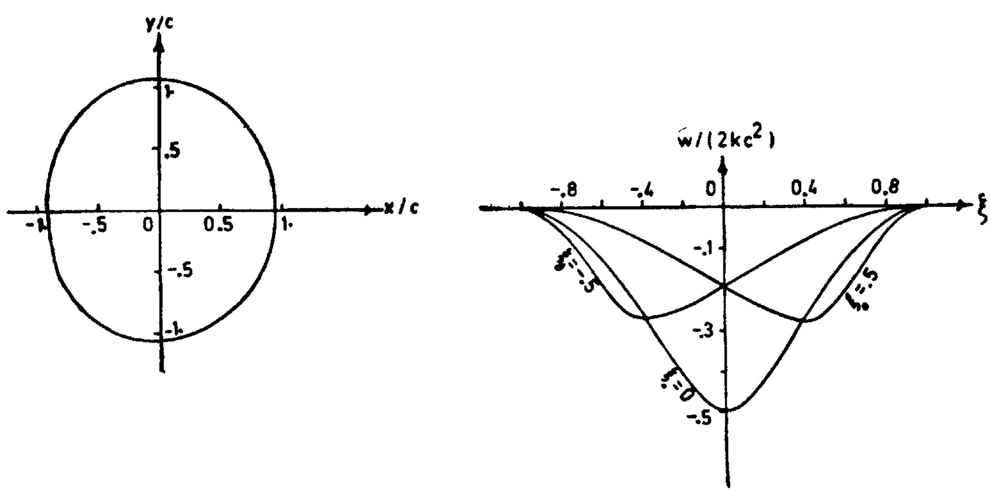

FIG. 9. $z = c\zeta(1 + 0.25\zeta)/(1 + 0.25\zeta + 0.0625\zeta^2)$, $\eta_0 = 0$.

$$\frac{w}{2kc^2} = \left|\frac{z - z_0}{c}\right|^2 \ln\left|\frac{1 - \bar{\zeta}_0\zeta}{\zeta - \zeta_0}\right| - \tfrac{1}{2}\, t(\rho_0)\, t(\rho). \qquad \ldots(24)$$

It is easily verified that the expressions obtained for w in all previous cases are symmetrical functions of z, z_0 and of ζ, ζ_0 i.e.,

$$w(z, z_0; \zeta, \zeta_0) = w(z_0, z; \zeta_0, \zeta)$$

so that the deflection at Q due to a transverse force at Q_0 is equal to the deflection at Q_0 due to a transverse force at Q.

References

Bassali, W. A. (1959). The analysis of singularly loaded and rigidly clamped thin elastic slabs with curvilinear boundaries—I. *Proc. Camb. phil. Soc.*, **55**, 121.

———— (1960). Some problems in the small deflexions of clamped thin isotropic plates. *Z. angew. Math. Mech.*, **40**, 493.

Bassali, W. A., and El-Sirafy, I. H. (1976). The analysis of thin clamped elastic plates under concentrated forces. *Bull. Calcutta math. Soc.*, **68**, 13–22.

Michell, J. H. (1901). Bending of a clamped circular plate. *Proc. Lond. math. Soc.*, **34**, 223.