Indian J. pure appl. Math., 12(9) : 1063–1068, September 1981

FINSLER SPACE WITH RECURRENT NEO-PSEUDO PROJECTIVE CURVATURE TENSOR

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(*Received* 17 *September* 1980; *after revision* 5 *March* 1981)

In the present paper we have studied some recurrent properties of neo-pseudo projective curvature tensor defined in a Finsler space by Singh and Singh (1979).

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n. The neo-pseudo projective deviation tensor $Q^i_j\,(x, \dot{x})$ is defined (Singh and Singh 1979) by

$$Q^i_j = pW^i_j + qT^i_j \qquad \text{...(1.1)}$$

where W^i_j and T^i_j are projective deviation tensor field, Pseudo deviation tensor field and p and q are scalars which are positively homogeneous of degree zero in $\dot{x}^i$.

The neo-pseudo projective curvature tensors Q^i_{jk} and Q^i_{jhk} are defined by (Singh and Singh 1979)

$$Q^i_{jk} = \tfrac{1}{3}\,(\dot{\partial}_j Q^i_k - \dot{\partial}_k Q^i_j) \qquad \text{...(1.2)}$$

$$Q^i_{jhk} = \dot{\partial}_j\, Q^i_{hk}\,. \qquad \text{...(1.3)}$$

The neo-pseudo projective curvature tensor satisfies the following identities

$$\left.\begin{aligned} & Q^i_{jhk} + Q^i_{hkj} + Q^i_{kjh} = 0 \\ & Q^i_{hk}\,\dot{x}^h = Q^i_k\,,\ \ Q^i_{jhk}\,\dot{x}^j = Q^i_{hk} \end{aligned}\right\} \qquad \text{...(1.4)}$$

which imply

$$\text{(a)}\ Q^i_{jhk}\,\dot{x}^j\dot{x}^h = Q^i_k\,,\quad \text{(b)}\ Q^i_j\,\dot{x}^j = 0. \qquad \text{...(1.5)}$$

Also we have a scalar function $Q(x, \dot{x})$ defined by

$$Q^i_i = q(n-1)\,T \quad [(W^i_i = 0,\ T^i_i = (n-1)\,T \text{ (Rund 1959)}]. \qquad ...(1.6)$$

Now it has been shown by Dubey and Singh (1979) that

$$W^i_j = H^i_j + T^i_j \qquad ...(1.7a)$$

where H^i_j is positively homogeneous of degree 1 in $\dot{x}^i$.

Hence from (1.1) we get, when $p = q = 1$

$$2\,W^i_j = H^i_j + Q^i_j \qquad ...(1.7b)$$

$$2\,W^i_{jk} = H^i_{jk} + Q^i_{jk} \qquad ...(1.7c)$$

$$2\,W^i_{jhk} = H^i_{jhk} + Q^i_{jhk}. \qquad ...(1.7d)$$

Here W^i_{jk} and W^i_{jhk} are projective curvature tensors.

2. Finsler Space with Recurrent Neo-pseudo Projective Curvature Tensor

We know that (Singh and Singh 1979) if Q^i_j is recurrent then projective deviation tensor and pseudo deviation tensor are proportional to each other.

i.e. $\qquad W^i_j = tT^i_j$,

where t is a scalar.

Thus from above and (1.1), we have

$$Q^i_j = sT^i_j \qquad ...(2.1a)$$

where $s = (pt + q)$ is any scalar and positively homogeneous of degree zero in $\dot{x}^i$.

The Finsler space is called Q-recurrent if

$$Q^i_{jhk(l)} = R_l Q^i_{jhk}, \qquad ...(2.1b)$$

where R_l is the recurrence vector field (Dubey and Singh 1979).

If $Q^i_{jhk(l)} = 0$, then the space will be called Q-symmetric and if $Q^i_{jhk} = 0$, it is called Q-flat.

Transvecting (2.1b) first by $\dot{x}^j$ and then by $\dot{x}^h$ and using (1.5a), we get

$$Q^i_{hk(l)} = R_l Q^i_{hk} \qquad ...(2.2)$$

$$Q^i_{k(l)} = R_l Q^i_k . \qquad \ldots(2.3)$$

Theorem 2.1 — The recurrence vector space R_l satisfies the following relation:

$$R_{l(m)} - R_{m(l)} = H^i_{lm} \dot{\partial}_i (\log T). \qquad \ldots(2.4)$$

PROOF : Differentiating (2.3) covariantly, we get

$$Q^i_{k(l)(m)} = (R_{l(m)} + R_{(l)} R_{(m)}) Q^i_k . \qquad \ldots(2.5)$$

Interchanging l and m in (2.5) and subtracting the new equation from (2.5), we get

$$Q^i_{k(l)(m)} - Q^i_{k(m)(l)} = (R_{l(m)} - R_{m(l)}) Q^i_k$$

using commutation formula [Rund 1959, (6.10a), p. 126] and (2.1a), we get

$$(R_{l(m)} - R_{m(l)}) Q^i_k = - (\dot{\partial}_\gamma Q^i_k) H^\gamma_{lm} + Q^\gamma_k H^i_{\gamma lm} - Q^i_\gamma H^\gamma_{klm}. \ldots(2.6)$$

Contracting i and k in the above equation, we get (2.4) due to (1.6).

The following theorems can be deduced from (1.7d).

Theorem 2.2 — If F_n statisfies the two of the following, it satisfies the third also:

(i) the space is projectively flat,

(ii) the space is H-flat,

(iii) the space is Q-flat.

Theorem 2.3 —If two of the following hold in F_n for the same recurrent vector field then third also holds:

(i) the space is projectively recurrent,

(ii) the space is H-recurrent, i.e., $H^i_{jhk(l)} = R_l H^i_{jhk}$,

(iii) the space is Q-recurrent.

Theorem 2.4 — If two of the following hold in F_n, then third also holds:

(i) the space is projectively symmetric,

(ii) the space is H-symmetric, i.e., $H^i_{jhk(l)} = 0$,

(iii) the space is Q-symmetric.

We have already seen that in a T-recurrent space the tensor Q^i_j and Q^i_{jk} are also recurrent. The converse part is given below.

Theorem 2.5 — Let Q^j_{hk} be recurrent in F_n, i.e.,

$$Q^j_{hk(l)} = R_l Q^j_{hk} \qquad \ldots(2.7)$$

then F_n will be Q-recurrent with same recurrence vector field if

$$(\dot{\partial}_i R_l)\ Q^j_{hk} = Q^m_{hk}\ G^j_{mil} + Q^j_{mh}\ G^m_{kil} - Q^j_{mk}\ G^m_{hil}. \qquad \ldots(2.8)$$

Proof : Differentiating (2.7) partially with respect to $\dot{x}^i$ and using commutation formula ((6.11b), p. 127, Rund (1959)), we get the result due to (2.1*a*) and (2.1*b*).

Theorem 2.6 — The recurrence vector field R_l is homogeneous of degree zero in $\dot{x}^i$.

Proof : Transvecting (2.8) with $\dot{x}^i$ and $\dot{x}^h$, we get

$$(\dot{x}^i \dot{\partial}_i R_l)\ Q^j_k = 0 \qquad \ldots(2.9)$$

because of $G^j_{mil}\ \dot{x}^m = 0$ and G^j_{mil} is completely symmetric in lower indices. Contracting j and k in (2.9) and noting the fact that Q does not vanish we have the statement.

Similarly the following theorem can be proved.

Theorem 2.7 — Q^j_{hk} is recurrent if

$$Q^j_k\ (\dot{\partial}_h R_l) - Q^j_h\ \dot{\partial}_k R_l + Q^r_h\ G^j_{krl} - Q^r_k\ G^j_{hrl} = 0. \qquad \ldots(2.10)$$

Theorem 2.8 — In projectively flat Q-recurrent space, we have

$$R_l Q^i_{hk} + R_h Q^i_{kl} + R_k Q^i_{lh} = 0. \qquad \ldots(2.11)$$

Proof : From (1.7*d*), we have

$$H^i_{jhk} + Q^i_{jhk} = 0.$$

Differentiating the above equation covariantly with respect to x^l, we get

$$H^i_{jhk(l)} + Q^i_{jhk(l)} = 0.$$

Taking the cyclic permutation in h, k, l and transvecting the result with $\dot{x}^j$, we have

$$(H^i_{jhk(l)} + H^i_{jkl(h)} + H^i_{jlh(k)})\, \dot{x}^j + (Q^i_{jhk(l)} + Q^i_{jkl(h)} + Q^i_{jlh(k)})\, \dot{x}^j = 0. \qquad ...(2.12)$$

The first part vanishes identically and hence we have

$$Q^i_{hk(l)} + Q^i_{kl(h)} + Q^i_{lh(k)} = 0.$$

(2.11) follows immediately from it by using (2.2).

3. Neo-pseudo Projective Recurrent Space of Second Order

Definition 3.1 — A Finsler space F_n whose neo-pseudo projective curvature tensor Q^i_{jhk} satisfies

$$Q^i_{jhk(l)(m)} = K_{lm}\, Q^i_{jhk} \qquad ...(3.1)$$

is called Neo-Pseudo projective recurrent space of second order or simply Q-birecurrent space with recurrence tensor K_{lm} (Singh and Singh 1979). If $K_{lm} = 0$, the space will be called Q-bisymmetric.

Contracting i and k in (3.1), we get

$$Q_{jh(l)(m)} = K_{lm} Q_{jh}. \qquad ...(3.2)$$

Theorem 3.1 — The recurrence tensor K_{lm} is non-symmetric.

Proof : From (3.2) we immediately have

$$Q_{jh(l)(m)} - Q_{jh(m)(l)} = K^*_{lm}\, Q_{jh} \qquad ...(3.3)$$

where

$$K^*_{lm} = K_{lm} - K_{ml}.$$

Transvecting (3.3) by $\dot{x}^j$, $\dot{x}^h$, we get

$$Q_{(l)(m)} - Q_{(m)(l)} = K^*_{lm}\, Q.$$

Using commutation formula [Rund (1959), (6.10), p. 126] and (2.1*a*), we get

$$QK^*_{lm} = -(\dot{\partial}_r Q)\, H^r_{lm} = (\dot{\partial}_r Q)\, H^r_{ml},$$

which implies

$$K^*_{lm} = \dot{\partial}_r (\log T)\, H^r_{ml}$$

which shows that $K_{lm} \neq K_{ml}$.

Theorem 3.2 — Every Q-recurrent space is Q-birecurrent.

PROOF : Differentiating (2.1b) covariantly with respect to x^m, we get

$$Q^i_{jhk(l)(m)} = K_{lm} Q^i_{jhk}$$

where $K_{lm} = R_{l(m)} - R_l R_m$.

From above equation it can be deduced that if Q-recurrent space is Q-bisymmetric, the recurrence vector is recurrent.

Theorem 3.3 — In a projectively flat Q-birecurrent space, we have

$$K_{lm} Q^i_{hk} + K_{hm} Q^i_{kl} + K_{km} Q^i_{lh} = 0. \quad ...(3.4)$$

The proof follows the pattern of (2·11).

Theorem 3.4 — If projectively flat Q-birecurrent space is also Q-recurrent then

$$K^*_{lm(n)} = R_n K^*_{lm}. \quad ...(3.5)$$

PROOF : Transvecting (3.1) with $\dot{x}^j$, we get

$$Q^i_{hk(l)(m)} = K_{lm} Q^i_{hk} \quad ...(3.6)$$

which gives

$$Q^i_{hk(l)(m)} - Q^i_{hk(m)(l)} = K^*_{lm} Q^i_{hk}.$$

Using commutation formula [Rund 1959, (6·10a), p. 126] and (2.1a), we have

$$(-\dot{\partial}_r Q^i_{hk}) H^r_{lm} + Q^r_{hk} H^i_{rlm} - Q^i_{rk} H^r_{hlm} - Q^i_{hr} H^r_{klm} = K^*_{lm} Q^i_{hk}.$$

Taking covariant derivation of the above equation and using (2.1b), we get

$$K^*_{lm(n)} Q^i_{hk} = K^*_{lm} R_n Q^i_{hk},$$

which yields (3.5) because $Q^i_{hk} \neq 0$.

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