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SOME COMMUTATION FORMULAE OF A COMPLEX MANIFOLD WITH DIRECTION DEPENDENT CONNECTIONS

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Rund (1967) introduced the concept of a n-dimensional complex manifold C_n depending on direction also. Later on, he (Rund 1972) introduced the concept of covariant differentiation with the help of a set of differentiable direction dependent connections $\Gamma^h_{jk}(\dot{z}, \dot{z}^*, z, z^*)$ together with its starred counterpart Γ^{h*}_{j*k*}. The purpose of the present paper is to obtain various commutation formulae involving partial derivatives with respect to $\dot{z}^h$ and $\dot{z}^{h*}$ and covariant derivatives with respect to z^h and z^{h*}. In the last article, we have derived some identities for the metric tensor based on the various commutation formulae obtained in the previous sections.

1. INTRODUCTION

Let X_{2n} be a $2n$-dimensional differentiable manifold referred to real coordinates (x^j, y^j) [Here, and in the sequel, all latin indices $j, h, k, \ldots$ vary from 1 to n, and the starred indices $j, h, k \ldots$ also take the same values]. To any point of X_{2n} with coordinates (x^j, y^j) we may assign complex coordinates

$$z^j = x^j + iy^j, z^{j*} = x^j - iy^j, (i = \sqrt{-1}).$$

With the help of above complex coordinates, we may define a point in a complex manifold of n-dimension, which is denoted by C_n. In such a space we assume the metric function $L(z^j, z^{j*}, \dot{z}^j, \dot{z}^{j*})$, where $\dot{z}^j = dz^j/dt$, $\dot{z}^{j*} = dz^{j*}/dt$, which satisfies the strong homogeneity condition $(\partial L/\partial \dot{z}^j)\, \dot{z}^j = (L/2)$, $(\partial L/\partial \dot{z}^{j*})\, \dot{z}^{j*} = (L/2)$. With the help of this metric function, the metric tensors are defined as follows (Rund 1967):

$$g_{hj} = \tfrac{1}{2}\, \dot{\partial}_h \dot{\partial}_j L^2, \quad g_{h_* j_*} = \tfrac{1}{2}\, \dot{\partial}_{h_*} \dot{\partial}_{j_*} L^2, \quad g_{h j_*} = \tfrac{1}{2}\, \dot{\partial}_h \dot{\partial}_{j_*} L^2,$$

where $\quad \dot{\partial}_h \equiv \partial/\partial \dot{z}^h$ and $\dot{\partial}_{h_*} \equiv \partial/\partial \dot{z}^{h*}$.

Assuming that the space C_n is endowed with a set of differentiable direction dependent connection $\Gamma^h_{jk}(z, z^*, \dot{z}, \dot{z}^*)$ together with its starred counterpart $\Gamma^{h*}_{j*k*}(z, z^*, \dot{z}, \dot{z}^*)$, Rund (1972) defined the process of covariant differentiation with respect to z^k by the relation

$$T^{jl*}_{mn*\,|\,k} = \partial_k T^{jl*}_{mn*} - \dot{\partial}_s T^{jl*}_{mn*} \Gamma^s_k + T^{tl*}_{mn*} \Gamma^j_{tk} - T^{jl*}_{tn*} \Gamma^t_{mk} \qquad \ldots(1.1)$$

and with respect to z^{k*} by the relation

$$T^{jl*}_{mn*\,|\,k*} = \partial_{k*} T^{jl*}_{mn*} - \dot{\partial}_{s*} T^{jl*}_{mn*} \Gamma^{s*}_{k*} + T^{jt*}_{mn*} \Gamma^{l*}_{t*k*} - T^{jl*}_{mt*} \Gamma^{t*}_{n*k*} \qquad \ldots(1.2)$$

where $\partial_k \equiv \partial/\partial z^k$, $\partial_{k*} \equiv \partial/\partial z^{k*}$ and we have written

$$\text{(a)}\ \Gamma^s_k = \Gamma^s_{rk}\, \dot{z}^r, \quad \text{(b)}\ \Gamma^{s*}_{k*} = \Gamma^{s*}_{r*k*}\, \dot{z}^{r*}. \qquad \ldots(1.3)$$

Now, differentiating (1.3*a*) with respect to $\dot{z}^h$ and (1.3*b*) with respect to $\dot{z}^{h*}$, we find,

$$\left.\begin{aligned} &\text{(a)}\ \Gamma^s_{hk} - \dot{\partial}_h \Gamma^s_k = -(\dot{\partial}_h \Gamma^s_{rk})\, \dot{z}^r \\ &\text{(b)}\ \Gamma^{s*}_{h*k*} - \dot{\partial}_{h*} \Gamma^{s*}_{k*} = -(\dot{\partial}_{h*} \Gamma^{s*}_{r*k*})\, \dot{z}^{r*}. \end{aligned}\right\} \qquad \ldots(1.4)$$

Rund (1972) has already shown that when the connection coefficients Γ^j_{hk} and Γ^{j*}_{h*k*} become metric (i.e., they statisfy Ricci's lemma), they are identical with the coefficients $G^j_{h,k}$ and $G^{j*}_{h*,k*}$ respectively, which are given as follows (Rund 1967):

$$G^j_{h,k} = \dot{\partial}_k g^{jl*}\, \partial_h g_{l*m}\, \dot{z}^m + g^{jl*}\, \partial_h g_{l*k} \qquad \ldots(1.5a)$$

$$G^{j*}_{h*,k*} = \dot{\partial}_{k*}\, g^{j*l}\, \partial_{h*}\, g_{lm*}\, \dot{z}^{m*} + g^{j*l}\, \partial_{h*}\, g_{lk*}. \qquad \ldots(1.5b)$$

2. Commutation Formulae Involving the Process $\dot{\partial}_k$ and $|\,k$

Let $T^j_m(z, z^*, \dot{z}, \dot{z}^*)$ be a tensor of contravariant order $(1+0)$ and covariant order $(1+0)$ in C_n, then according to (1.1) its covariant derivative with respect to z^k is given by the relation

$$T^j_{m\,|\,k} = \partial_k T^j_m - \dot{\partial}_s T^j_m \Gamma^s_k + T^t_m \Gamma^j_{tk} - T^j_t \Gamma^t_{mk} \qquad \ldots(2.1)$$

and of $\dot{\partial}_h T^j_m$ with respect to z^k by the equation

$$(\dot{\partial}_h T^j_m)_{\,|\,k} = \partial_k \dot{\partial}_h T^j_m - \dot{\partial}_s \dot{\partial}_h T^j_m \Gamma^s_k + \dot{\partial}_h T^t_m \Gamma^j_{tk} - \dot{\partial}_t T^j_m \Gamma^t_{hk} - \dot{\partial}_h T^j_t \Gamma^t_{mk}\,. \qquad \ldots(2.2)$$

Now, differentiating (2.1) with respect to $\dot{z}^h$ and subtracting (2.2) from the equation thus obtained and using (1.4a) we find

$$\dot{\partial}_h(T^j_{m\,|\,k}) - (\dot{\partial}_h T^j_m)_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^j_m) + T^t_m \dot{\partial}_h \Gamma^j_{tk} - T^j_t\, \dot{\partial}_h \Gamma^t_{mk}\,. \quad \ldots(2.3)$$

Proceeding as above, for the tensors T^{l*}_m, T^j_{n*} and T^{l*}_{n*} we get

$$\dot{\partial}_h(T^{l*}_{m\,|\,k}) - (\dot{\partial}_h T^{l*}_m)_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^{l*}_m) - T^{l*}_t\, \dot{\partial}_h \Gamma^t_{mk} \quad \ldots(2.4)$$

$$\dot{\partial}_h(T^j_{n*\,|\,k}) - (\dot{\partial}_h T^j_{n*})_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^j_{n*}) + T^t_{n*}\, \dot{\partial}_h \Gamma^j_{tk} \quad \ldots(2.5)$$

$$\dot{\partial}_h(T^{l*}_{n*\,|\,k}) - (\dot{\partial}_h T^{l*}_{n*})_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^{l*}_{n*}). \quad \ldots(2.6)$$

Equations (2.3), (2.4), (2.5) and (2.6) are commutation formulae for a mixed tensor of order two of various types.

Next, let $T^{jl}_{mn}(z, z^*, \dot{z}, \dot{z}^*)$ be a tensor of contravariant valency $(2 + 0)$ and covariant valency $(2 + 0)$ in C_n, then according to (1.1), its covariant derivative with respect to z^k is given by the relation

$$T^{jl}_{mn\,|\,k} = \partial_k T^{jl}_{mn} - \dot{\partial}_s T^{jl}_{mn}\Gamma^s_k + T^{tl}_{mn}\Gamma^j_{tk} + T^{jt}_{mn}\Gamma^l_{tk} - T^{jl}_{tn}\Gamma^t_{mk} - T^{jl}_{mt}\Gamma^t_{nk} \quad \ldots(2.7)$$

and that of $\dot{\partial}_h T^{jl}_{mn}$ with respect to z^k by the relation

$$(\dot{\partial}_h T^{jl}_{mn})_{|\,k} = \partial_k\dot{\partial}_h T^{jl}_{mn} - \dot{\partial}_s\dot{\partial}_h T^{jl}_{mn}\Gamma^s_k + \dot{\partial}_h T^{tl}_{mn}\Gamma^j_{tk} + \dot{\partial}_h T^{jt}_{mn}\Gamma^l_{tk}$$
$$- \dot{\partial}_t T^{jl}_{mn}\Gamma^t_{hk} - \dot{\partial}_h T^{jl}_{tn}\Gamma^t_{mk} - \dot{\partial}_h T^{jl}_{mt}\Gamma^t_{nk}. \quad \ldots(2.8)$$

On subtracting (2.8) from the equation obtained after differentiating (2.7) with respect to $\dot{z}^h$, by virtue of (1.4a), we find

$$\dot{\partial}_h(T^{jl}_{mn\,|\,k}) - (\dot{\partial}_h T^{jl}_{mn})_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^{jl}_{mn}) + T^{tl}_{mn}\,\dot{\partial}_h \Gamma^j_{tk}$$
$$+ T^{jt}_{mn}\dot{\partial}_h \Gamma^l_{tk} - T^{jl}_{tn}\,\dot{\partial}_h \Gamma^t_{mk} - T^{jl}_{mt}\dot{\partial}_h \Gamma^t_{nk}. \quad (2.9)$$

For the tensors T^{jl*}_{mn*}, T^{j*l*}_{m*n*} a similar process yields

$$\dot{\partial}_h(T^{jl*}_{mn*\,|\,k}) - (\dot{\partial}_h T^{jl*}_{mn*})_{|\,k} = -\dot{z}^r(\dot{\partial}_h \Gamma^s_{rk})\,(\dot{\partial}_s T^{jl*}_{mn*}) + T^{tl*}_{mn*}\,\dot{\partial}_h \Gamma^j_{tk}$$
$$- T^{jl*}_{tn*}\,(\dot{\partial}_h \Gamma^t_{mk}) \quad \ldots(2.10)$$

$$\dot{\partial}_h(T^{j*l*}_{m*n*\,|\,k}) - (\dot{\partial}_h T^{j*l*}_{m*n*})_{\,|\,k} = -\dot{z}^r(\dot{\partial}_h\Gamma^{s}_{rk})\,(\dot{\partial}_s T^{j*l*}_{m*n*}). \qquad \ldots(2.11)$$

Equations (2.9), (2.10) and (2.11) are commutation formulae for a mixed tensor of order 4 of various types. All the above commutation formulae obtained in this section clearly suggest that the commutation formulae for a general tensor $T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*}$ $(z, z^*, \dot{z}, \dot{z}^*)$ of contravariant valency $(p + q)$ and covariant valency $(a + b)$ is given by the equation

$$\dot{\partial}_h\left(T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*\,|\,k}\right) - \left(\dot{\partial}_h T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*}\right)_{|\,k}$$

$$= -\dot{z}^r(\dot{\partial}_h\Gamma^{s}_{rk})\left(\dot{\partial}_s T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*}\right)$$

$$+\sum_{\alpha=1}^{p} T^{j_1\ldots j_{\alpha-1} t j_{\alpha+1}\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots\ldots\ldots\ldots m_a n_1^*\ldots n_b^*}(\dot{\partial}_h\Gamma^{j_\alpha}_{tk})$$

$$-\sum_{\beta=1}^{a} T^{j_1\ldots\ldots\ldots\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_{\beta-1} t m_{\beta+1}\ldots m_a n_1^*\ldots n_b^*}(\dot{\partial}_h\Gamma^{t}_{m_\beta k}) \qquad \ldots(2.12)$$

3. Commutation Formulae Involving the Process $\dot{\partial}_{k*}$ and $|\,k^*$

Let T^{j*}_{m*} $(z, z^*, \dot{z}, \dot{z}^*)$ be a tensor of contravariant order $(0 + 1)$ and covariant order $(0 + 1)$, then according to (1.2), its covariant derivative with respect to z^{k*} is given by the relation

$$T^{j*}_{m*\,|\,k*} = \partial_{k*}T^{j*}_{m*} - \dot{\partial}_{s*}T^{j*}_{m*}\,\Gamma^{s*}_{k*} + T^{t*}_{m*}\,\Gamma^{j*}_{t*k*} - T^{j*}_{t*}\,\Gamma^{t*}_{m*k*} \qquad \ldots(3.1)$$

and that of $\dot{\partial}_{h*}T^{j*}_{m*}$ with respect to z^{k*} is given by the relations

$$(\dot{\partial}_{h*}T^{j*}_{m*})_{\,|\,k*} = \partial_{k*}\dot{\partial}_{h*}T^{j*}_{m*} - \dot{\partial}_{s*}\dot{\partial}_{h*}T^{j*}_{m*}\,\Gamma^{s*}_{k*} + \dot{\partial}_{h*}T^{t*}_{m*}\,\Gamma^{j*}_{t*k*}$$

$$- \dot{\partial}_{t*}T^{j*}_{m*}\,\Gamma^{t*}_{h*k*} - \dot{\partial}_{h*}T^{j*}_{t*}\,\Gamma^{t*}_{m*k*}. \qquad \ldots(3.2)$$

Taking the difference of (3.2) and the equation obtained after differentiating (3.1) with respect to $\dot{z}^{h*}$ and using (1.4*b*) we get

$$\dot{\partial}_{h*}(T^{j*}_{m*\,|\,k*}) - (\dot{\partial}_{h*}T^{j*}_{m*})_{\,|\,k*} = -\dot{z}^{r*}(\dot{\partial}_{h*}\Gamma^{s*}_{r*k*})\,(\dot{\partial}_{s*}T^{j*}_{m*})$$

$$+ T^{t*}_{m*}\dot{\partial}_{h*}\,\Gamma^{j*}_{t*k*} - T^{j*}_{t*}\,\dot{\partial}_{h*}\Gamma^{t*}_{m*k*}. \qquad \ldots(3.3)$$

A process similar to above yields the following commutation formulae for the tensors T^{j*}_{m}, T^{j}_{m*} and T^{j}_{m} of order 2 of various types

$$\dot{\partial}_{h*}(T^{j*}_{m\,|\,k*}) - (\dot{\partial}_{h*}T^{j*}_{m})_{|\,k*} = -\dot{z}^{r*}(\dot{\partial}_{h*}\Gamma^{s*}_{r*k*})(\dot{\partial}_{s*}T^{j*}_{m}) + T^{t*}_{m}\dot{\partial}_{h*}\Gamma^{j*}_{t*k*} \qquad \ldots(3.4)$$

$$\dot{\partial}_{h*}(T^{j}_{m*\,|\,k*}) - (\dot{\partial}_{h*}T^{j}_{m*})_{|\,k*} = -\dot{z}^{r*}(\dot{\partial}_{h*}\Gamma^{s*}_{r*k*})(\dot{\partial}_{s*}T^{j}_{m*}) - T^{j}_{t*}\dot{\partial}_{h*}\Gamma^{t*}_{m*k*} \qquad \ldots(3.5)$$

$$\dot{\partial}_{h*}(T^{j}_{m\,|\,k*}) - (\dot{\partial}_{h*}T^{j}_{m})_{|\,k*} = -\dot{z}^{r*}(\dot{\partial}_{h*}\Gamma^{s*}_{r*k*})(\dot{\partial}_{s*}T^{j}_{m}). \qquad \ldots(3.6)$$

Comparing these equations with the results obtained in §2, we see that eqns. (3.3), (3.4), (3.5) and (3.6) are starred counterparts of the equations (2.3), (2.4), (2.5) and (2.6) respectively. In analogy to this, we have the commutation formulae involving partial derivative with respect to $\dot{z}^{k*}$ and covariant derivative with respect to z^{k*} for a general tensor $T^{j_1\ldots j_p l_1^*\ldots\ldots\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots\ldots n\ldots_b^*}$ of contravariant valency $(p+q)$ and covariant valency $(a+b)$, which is given by the relation

$$\begin{aligned}
&\dot{\partial}_{h*}\left(T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*\,|\,k^*}\right) - \left(\dot{\partial}_{h*}T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*}\right)_{|\,k^*} \\
&\quad = -\dot{z}^{r*}\left(\dot{\partial}_{h*}\Gamma^{s*}_{r*k*}\right)\left(\dot{\partial}_{s*}T^{j_1\ldots j_p l_1^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_b^*}\right) \\
&\quad + \sum_{\alpha=1}^{q} T^{j_1\ldots j_p l_1^*\ldots l_{\alpha-1}^* t^* l_{\alpha+1}^*\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots\ldots\ldots n_b^*}\left(\dot{\partial}_{h*}\Gamma^{l_\alpha^*}_{t*k*}\right) \\
&\quad - \sum_{\beta=1}^{b} T^{j_1\ldots j_p l_1^*\ldots\ldots\ldots l_q^*}_{m_1\ldots m_a n_1^*\ldots n_{\beta-1}^* t^* n_{\beta+1}^*\ldots n_b^*}\left(\dot{\partial}_{h*}\Gamma^{t*}_{n_\beta^* k*}\right). \qquad \ldots(3.7)
\end{aligned}$$

4. Commutation Formulae Involving the Process $\dot{\partial}_k$ and $|\,k^*$ and Their Starred Counterparts

According to (1.2) the covariant derivative of $\dot{\partial}_h T^{j*}_{m*}$ with respect to z^{k*} is given by the relation

$$(\dot{\partial}_h T^{j*}_{m*})_{|\,k*} = \partial_{k*}\dot{\partial}_h T^{j*}_{m*} - \dot{\partial}_{s*}\dot{\partial}_h T^{j*}_{m*}\,\Gamma^{s*}_{k*} + \dot{\partial}_h T^{t*}_{m*}\,\Gamma^{j*}_{t*k*} - \dot{\partial}_h T^{j*}_{t*}\,\Gamma^{t*}_{m*k*}$$

which, on subtraction from the equation obtained after differentiating (3.1) with respect to $\dot{z}^h$, yields

$$\dot{\partial}_h(T^{j*}_{m*\,|\,k*}) - (\dot{\partial}_h T^{j*}_{m*})_{\,|\,k*} = -\,(\dot{\partial}_{s*} T^{j*}_{m*})\,(\dot{\partial}_h \Gamma^{s*}_{k*}) + T^{t*}_{m*}\dot{\partial}_h \Gamma^{j*}_{t*k*} - T^{j*}_{t*}\dot{\partial}_h \Gamma^{t*}_{m*k*}. \quad \ldots(4.1)$$

For the tensors T^{j}_{m}, T^{j*}_{m} and T^{j}_{m*}, a similar process yields the following equations

$$\dot{\partial}_h(T^{j}_{m\,|\,k*}) - (\dot{\partial}_h T^{j}_{m})_{\,|\,k*} = -\,(\dot{\partial}_{s*} T^{j}_{m})\,(\dot{\partial}_h \Gamma^{s*}_{k*}) \quad \ldots(4.2)$$

$$\dot{\partial}_h(T^{j*}_{m\,|\,k*}) - (\dot{\partial}_h T^{j*}_{m})_{\,|\,k*} = -\,(\dot{\partial}_{s*} T^{j*}_{m})\,(\dot{\partial}_h \Gamma^{s*}_{k*}) + T^{t*}_{m}\,\dot{\partial}_h \Gamma^{j*}_{t*k*} \quad \ldots(4.3)$$

$$\dot{\partial}_h(T^{j}_{m*\,|\,k*}) - (\dot{\partial}_h T^{j}_{m*})_{\,|\,k*} = -\,(\dot{\partial}_{s*} T^{j}_{m*})\,(\dot{\partial}_h \Gamma^{s*}_{k*}) - T^{j}_{t*}\dot{\partial}_h \Gamma^{t*}_{m*k*}. \quad \ldots(4.4)$$

Equations (4.1), (4.2), (4.3) and (4.4) are commutation formulae for a mixed tensor of order two of various types. As, in the last two articles, it is evident from the above relations that the commutation formula involving partial derivative with respect to $\dot{z}^k$ and covariant derivative with respect to z^{k*} for a general tensor

$$T^{j_1\ldots j_p l^*_1\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots n^*_b}\ (z, z^*, \dot{z}, \dot{z}^*)$$

of contravariant valency $(p + q)$ and covariant valency $(a + b)$ is given by the following equation

$$\begin{aligned}\dot{\partial}_h\Big(T^{j_1\ldots j_p l^*_1\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots n^*_b\,|\,k^*}\Big) - \Big(\dot{\partial}_h T^{j_1\ldots j_p l^*_1\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots n^*_b}\Big)_{\,|\,k^*} &= -\Big(\dot{\partial}_{s*} T^{j_1\ldots j_p l^*_1\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots n^*_b}\Big)\Big(\dot{\partial}_h \Gamma^{s*}_{k*}\Big)\\ &+ \sum_{\alpha=1}^{q} T^{j_1\ldots j_p l^*_1\ldots l^*_{\alpha-1} t^* l^*_{\alpha+1}\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots\ldots\ldots n^*_b}\Big(\dot{\partial}_h \Gamma^{l^*_\alpha}_{t*k*}\Big)\\ &- \sum_{\beta=1}^{b} T^{j_1\ldots j_p l^*_1\ldots\ldots\ldots\ldots l^*_q}_{m_1\ldots m_a n^*_1\ldots n^*_{\beta-1} t^* n^*_{\beta+1}\ldots n^*_b}\Big(\dot{\partial}_h \Gamma^{t*}_{n^*_\beta k^*}\Big). \quad \ldots(4.5)\end{aligned}$$

Next, we consider the commutation formulae involving partial derivatives with respect to $\dot{z}^{h*}$ and covariant derivative of type (1.1). The covariant derivative of $\dot{\partial}_{h*} T^{j}_{m}$ with respect to z^k is given by the relation

$$(\dot{\partial}_{h*} T^{j}_{m})_{\,|\,k} = \partial_k \dot{\partial}_{h*} T^{j}_{m} - \dot{\partial}_s \dot{\partial}_{h*} T^{j}_{m} \Gamma^{s}_{k} + \dot{\partial}_{h*} T^{t}_{m} \Gamma^{j}_{tk} - \dot{\partial}_{h*} T^{j}_{t}\, \Gamma^{t}_{mk}\,.$$

On subtracting this equation from the equation obtained after differentiating (2.1) with respect to $\dot{z}^{h*}$, we get

$$\dot{\partial}_{h*}(T^j_{m\,|\,k}) - (\dot{\partial}_{h*}T^j_m)_{|\,k} = -(\dot{\partial}_s T^j_m)(\dot{\partial}_{h*}\Gamma^s_k) + T^t_m \dot{\partial}_{h*}\Gamma^j_{tk} - T^j_t \dot{\partial}_{h*}\Gamma^t_{mk}. \quad \text{...(4.6)}$$

A similar process yields the following equations

$$\dot{\partial}_{h*}(T^{j*}_{m*\,|\,k}) - (\dot{\partial}_{h*}T^{j*}_{m*})_{|\,k} = -(\dot{\partial}_s T^{j*}_{m*})(\dot{\partial}_{h*}\Gamma^s_k) \quad \text{...(4.7)}$$

$$\dot{\partial}_{h*}(T^j_{m*\,|\,k}) - (\dot{\partial}_{h*}T^j_{m*})_{|\,k} = -(\dot{\partial}_s T^j_{m*})(\dot{\partial}_{h*}\Gamma^s_k) + T^t_{m*}\dot{\partial}_{h*}\Gamma^j_{tk} \quad \text{...(4.8)}$$

$$\dot{\partial}_{h*}(T^{j*}_{m\,|\,k}) - (\dot{\partial}_{h*}T^{j*}_m)_{|\,k} = -(\dot{\partial}_s T^{j*}_m)(\dot{\partial}_{h*}\Gamma^s_k) - T^{j*}_t \dot{\partial}_{h*}\Gamma^t_{mk}. \quad \text{...(4.9)}$$

Equations (4.6), (4.7), (4.8) and (4.9) are the commutation formulae for a mixed tensor of order 2 of various types, and, in fact, these equations are the starred counterparts of (4.1), (4.2), (4.3) and (4.4) respectively. Hence it is evident that the commutation formula involving partial derivative with respect to $\dot{z}^{h*}$ and covariant derivative of type (1.1) for a general tensor $T^{j_1 \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots m_a n^*_1 \dots n^*_b}(z, z^*, \dot{z}, \dot{z}^*)$ of contravariant order $(p+q)$ and covariant order $(a+b)$ is given by the equation

$$\dot{\partial}_{h*}\left(T^{j_1 \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots m_a n^*_1 \dots n^*_b\,|\,k}\right) - \left(\dot{\partial}_{h*}T^{j_1 \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots m_a n^*_1 \dots n^*_b}\right)_{|\,k}$$

$$= -\left(\dot{\partial}_s T^{j_1 \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots m_a n^*_1 \dots n^*_b}\right)\left(\dot{\partial}_{h*}\Gamma^s_k\right)$$

$$+ \sum_{\alpha=1}^{p} T^{j_1 \dots j_{\alpha-1} t j_{\alpha+1} \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots \dots \dots \dots m_a n^*_1 \dots n^*_b}\left(\dot{\partial}_{h*}\Gamma^{j_\alpha}_{tk}\right)$$

$$- \sum_{\beta=1}^{a} T^{j_1 \dots \dots \dots \dots j_p l^*_1 \dots l^*_q}_{m_1 \dots m_{\beta-1} t m_{\beta+1} \dots m_a n^*_1 \dots n^*_b}\left(\dot{\partial}_{h*}\Gamma^t_{m_\beta k}\right). \quad \text{...(4.10)}$$

5. Some Deductions

So far, we have assumed that our manifold was endowed with an arbitrary set of connection coefficients. In this section, we shall deal with the metric case. Rund (1972) showed that in such case the connection coefficients Γ^j_{hk} and Γ^{j*}_{h*k*} are identical with $G^{j}_{h,k}$ and $G^{j*}_{h*,k}$ given by (1.5*a*) and (1.5*b*) respectively and the covariant derivatives of the component g_{jk_*} of metric tensor vanish, while the covariant derivatives of the components g_{jl} and $g_{j_*l_*}$ do not vanish, i.e.

$$\text{(a)}\quad g_{j_*k\,|\,h} = 0 \qquad \text{(b)}\quad g_{jk_*\,|\,h_*} = 0 \qquad \ldots(5.1)$$

$$g_{jm\,|\,k} = \tfrac{1}{2}\, C_{m,jh_*\,|\,k}\, \dot{z}^{h*},\ \text{conj} \qquad \ldots(5.2)$$

$$g_{jm\,|\,k_*} = \tfrac{1}{2}\, C_{m,jh_*\,|\,k_*}\, \dot{z}^{h*},\ \text{conj} \qquad \ldots(5.3)$$

where we have used the notation $C_{m,jh_*} = \dot{\partial}_m g_{jh_*}$ and the word 'Conj' means that when starred indices are unstarred and unstarred indices are starred, a similar relation holds good. For the sake of brevity, we shall use the following notations

$$C_{h,jm} = \dot{\partial}_h g_{jm},\ C_{l_*,j_*r} = \dot{\partial}_{l_*} g_{j_*r},\ C_{h,j_*l_*} = \dot{\partial}_h g_{j_*l_*},$$

$$G^{j}_{mh,k} = \dot{\partial}_m G^{j}_{h,k},\ G^{j*}_{m*h*,k*} = \dot{\partial}_{m_*} G^{j*}_{h*,k*}\,.$$

In such case, the equations (1.4*a*) and (1.4*b*) reduces to the form

$$\text{(a)}\quad (\dot{\partial}_h G^{s}_{r,k})\, \dot{z}^{r} = 0, \qquad \text{(b)}\quad (\dot{\partial}_{h_*} G^{s*}_{r*,k*})\, \dot{z}^{r*} = 0. \qquad \ldots(5.4)$$

Also, the following identities will be frequently used

$$\text{(a)}\quad C_{s,j_*r}\, \dot{z}^{r} = 0, \qquad \text{(b)}\quad C_{h,tl_*}\, \dot{z}^{l*} = 2g_{ht}. \qquad \ldots(5.5)$$

On applying the relation (2.12) to g_{jl_*} and using (5.1*a*) and (5.4*a*), we find that

$$C_{m,jl_*\,|\,k} = g_{tl_*} G^{t}_{mj,k} \qquad \ldots(5.6)$$

whereas, if we apply the commutation formulae (2.12) to the tensor g_{jm} and use the relations (5.2), (5.4) (a) and (5.5) (b), after some simplification, we find that

$$C_{h,jm\,|\,k} = g_{tm}\, G^{t}_{hj,k} + g_{th} G^{t}_{mj,k} + g_{jt} G^{t}_{hm,k}$$
$$+ \tfrac{1}{2}\, g_{tl_*} \dot{z}^{l*} \dot{\partial}_h G^{t}_{mj,k}. \qquad \ldots(5.7)$$

Applying again equation (2.12) to the tensor $g_{j_*l_*}$ and using (5.2) 'Conj' and (5.4*a*), we find that

$$\dot{\partial}_h (C_{l_*,j_*r\,|\,k})\, \dot{z}^{r} + C_{l_*,j_*h\,|\,k} = 2C_{h,j_*l_*\,|\,k}. \qquad \ldots(5.8)$$

But if we apply (4.10) to the tensor g_{j_*t}, we have

$$C_{l_*,j_*r\,|\,k} = C_{s,j_*r} (\dot{\partial}_{l_*} G^{s}_{k}) - g_{j_*t} (\dot{\partial}_{l_*} G^{t}_{rk}). \qquad \ldots(5.9)$$

Substituting from (5.9) in (5.8) and simplifying with the help of (5.5*a*) we find

$$C_{h,j_*l_*|k} = -\tfrac{1}{2}\,[\, g_{j_*t}\; \dot{\partial}_{l_*} G^t_{h,k} + C_{h,j_*t}\; \dot{\partial}_{l_*} G^t_k \,]. \qquad \ldots(5.10)$$

In the end, it can be remarked that various identities for the covariant derivatives of directional derivatives of components of metric tensor (i.e. g_{jl}, g_{j_*l}, $g_{j_*l_*}$) based on the commutation formulae (3.7), (4.5), and (4.10) can be easily obtained in a way similar to (5.6), (5.7) and (5.10).

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