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A NOTE ON THE NUMBER OF COPRIME INTEGER SOLUTIONS OF $y^2 = ax^3 + k$

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Let $N'(a, k)$ and $N'(k)$ be the numbers of coprime integer solutions of $y^2 = ax^3 + k$ and $y^2 = x^3 + k$ respectively. In this paper we prove that $\limsup_{k\to\infty} N'(a, k) \geqslant 6$ holds for odd integer a, and give another proof of $\limsup_{k\to\infty} N'(k) \geqslant 8$.

The Diophantine equation $y^2 = x^3 + k$ has played a fundamental role in the development of number theory (see Mordell (1969)). Let $N'(k)$ be the number of coprime integer solutions of $y^2 = x^3 + k$. Mohanty (1973, 1975) proved that $\limsup_{k\to\infty} N'(k) \geqslant 6$, Stephens (1975) proved that $\limsup_{k\to\infty} N'(k) \geqslant 8$. In this paper we generalize Mohanty's result to the Diophantine equation $y^2 = ax^3 + k$, where a is an odd integer, and give another proof of Stephens' result.

§1. Consider the following polynomials:

$$x_1 = 2t, \qquad y_1 = 2t^2 + a^2;$$

$$x_2 = -a, \qquad y_2 = 2t^2 - 2at;$$

$$x_3 = -2t + 2a, \quad y_3 = 2t^2 - 4at + 3a^2.$$

It is easy to check that $(x_i, \pm y_i)$ $(i = 1, 2, 3)$ are solutions of the Diophantine equation $y^2 = ax^3 + k$, where $k = 4t^4 - 8at^3 + 4a^2t^2 + a^4$. If a is an odd integer and t satisfies $(t, a) = 1$, then $(x_i, y_i) = 1$, $i = 1, 2, 3$. To save space we only check that $(x_3, y_3) = 1$.

$$\begin{aligned}(x_3, y_3) &= (-2t + 2a, 2t^2 - 4at + 3a^2) = (2(t - a), -2at + 3a^2)\\ &= (t - a, -2at + 3a^2) = (t - a, a^2) = (t-a, a)\\ &= (t, a) = 1.\end{aligned}$$

Hence if we denote the number of coprime integer solutions of $y^2 = ax^3 + k$ by $N'(a, k)$, then we have:

Theorem 1 — $\limsup_{k\to\infty} N'(a, k) \geqslant 6$ holds for any odd integer a.

§2. Consider the following polynomials :

$$x_1 = 2t, \qquad y_1 = 3t^2 - 3t + 3;$$

$$x_2 = -4t + 6, \qquad y_2 = 3t^2 - 15t + 15;$$

$$x_3 = 2t - 2, \qquad y_3 = 3t^2 - 3t - 1;$$

$$x_4 = 9t^4 - 18t^3 + 15t^2 - 10t + 3,$$

$$y_4 = 27t^6 - 81t^5 + 108t^4 - 99t^3 + 63t^2 - 24t + 6.$$

It is easy to check that $(x_i, \pm y_i)$ $(i = 1, 2, 3, 4)$ are solutions of the Diophantine equation $y^2 = x^3 + k$, where $k = 9t^4 - 26t^3 + 27t^2 - 18t + 9$. If $(t, 3) = 1$, it can be easily seen that $(x_i, y_i) = 1$ for $i = 1, 2, 3, 4$ by applying Euclid's algorithm, hence we have:

Theorem 2 — $\limsup_{k\to\infty} N'(k) \geqslant 8$.

§3. It is worth mentioning that our k is a polynomial of degree 4, while Mohanty and Stephens' k is a polynomial of degree 6. Similar to Stephens (1975), we may raise the following:

Problem 1 — Does there exist a polynomial $k(t)$ with integral coefficients and degree 4, such that there are eight solutions of $y(t)^2 = x(t)^3 + k(t)$ with $\deg x(t) \leqslant 2$ and $\deg y(t) \leqslant 3$?

Another natural problem is:

Problem 2 — Does $\limsup_{k\to\infty} N'(a, k) \geqslant 6$ hold for even integer a ?

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