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SOME COMMENTS ON THE PAPER BY KARADE AND KUMBHARE ENTITLED "ALTERNATIVE APPROACH TO THE SIMPLEX METHOD"

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The objective of this note is to show that the method of inserting a new vector from the non-basic to the basic group while jumping from one extreme point to an adjacent extreme point of the convex polygon of a L.P. problem discussed by Karade and Kumbhare (1980) is not new. This procedure is sometimes faster than the generally used simplex algorithm but it cannot always resolve a cyclic L.P. problem as claimed by them.

1. INTRODUCTION

The central problem of linear programming is to minimize a linear expression of the type

$$z = c_j x_j \qquad \text{...(1)}$$

subject to the constraints

$$a_{ij} x_j = b_i \qquad \text{...(2)}$$

and

$$x_j \geqslant 0;\ i = 1, 2, .., m, j = 1, 2, ..., n \qquad \text{...(3)}$$

where $[a_{ij}]$ is a $m \times n$ matrix, $[b_i]$, $[x_j]$ are two $m \times 1$ and $n \times 1$ column matrices, $[c_j]$ is a row matrix of order $1 \times n$. The columns of the matrix $[a_{ij}]$ are denoted by $P_1, P_2, ... P_n$.

2. INITIAL BASIS

As the constraint equations are a set of m equations in n unknowns, this system can be solved in terms of the $(n - m)$ remaining variables. Without any loss of generality, the first m variables $(x_1, x_2, x_3 ..., x_m)$ can be taken to form the basic feasible solution. These m variables can be easily obtained by elimination procedure from eqn. (2). These are given by

$$x_i = x_{i0} - x_{im+1} x_{m+1} - x_{im+2} x_{m+2} - \ldots - x_{in} x_n,\ (i = 1, 2, \ldots m). \qquad \text{...(4)}$$

Substituting the non-basic variables x_j $(j = m + 1, m + 2, ..., n)$ equal to zero in the above equation, the usual non-degenerate basic feasible solution (coordinates of the extreme point) are obtained as

$$X_0 = (x_{10}, x_{20}, \ldots x_{m0}, 0, 0, \ldots, 0).$$

Eliminating these m variables from eqn. (1) we get

$$z = z_0 - \sum_{j=m+1}^{n} (z_j - c_j)\, x_j \qquad \ldots(5)$$

where z_0 is the value of the objective function corresponding to the basic feasible solution X_0. c_j are the cost coefficients and z_j has its own usual value.

Since all $x_j \geqslant 0$, from the above equation we find that if for the basic feasible solution X_0 all $(z_j - c_j) \leqslant 0$ then an increase above zero of any non-basic variable x_j with $(z_j - c_j) < 0$ would increase z, while if some $(z_j - c_j) > 0$ an increase of the corresponding non-basic variable would decrease z.

The problem being one of minimization, we are naturally interested to minimize z.

3. Improving the Basis

In order to reduce z, we generally choose the vector* corresponding to which $(z_j - c_j)$ is maximum positive as the entering vector to the basic group. Karade and Kumbhare (1980) instead of choosing the entering vector as above choose the vector for which $\theta_j(z_j - c_j)$ is maximum positive.

where $\qquad \theta = \min\limits_{i} \left(\dfrac{x_{i0}}{x_{ij}}\right) > 0,\ x_{ij} > 0$ for fixed j.

This sort of choice for the entering vector is not at all new and cannot be claimed as original.

In this connection we wish to quote a few lines from the book by Gass (1964, p. 72):

"As Dantzig points out, the number of iterations, i.e., the number of basis changes, necessary to obtain a minimum solution can, in general be greatly reduced by not selecting at random any vector P_j, with its $(z_j - c_j) > 0$, but by selecting the one which gives the greatest immediate decrease in the value of the objective function. The vector P_j should then be the one which corresponds to the $\max\limits_{j} \theta_0 (z_j - c_j)$ where for each j, θ_0 is given by

$$\theta_0 = \min_{i} \left(\frac{x_{i0}}{x_{ij}}\right) > 0.$$

*In searching for a new vector to enter the basis we can theoretically select any vector whose corresponding $(z_j - c_j) > 0$.

If there are a number of j for which $(z_j - c_j) > 0$, the above rule is rather complicated to apply. A much simpler criterion for selecting the vector to be introduced is to select the one which corresponds to the max $(z_j - c_j)$".
From this, it is clear that the above-mentioned way of choosing the entering vector to the basis is not new and it has been discovered long ago by Dantzig (1951).

The other point in which we wish to differ from Karade and Kumbhare (1980) is that, this way of choosing the entering vector to the basis cannot always resolve cycling *L-P* problems. Karade and Kumbhare (1980) in their paper have solved the well-known cycling problem due to Beale (1955) in three iterations. Though this *L-P* problem completes one cycle at the sixth iteration if the normal selection rules for the entering vector are applied, it can be solved and cycling avoided if right choices are made while choosing the entering vector or the one to be dropped from the basis, vide, Garvin (1960). The use of alternative selection rules for the entering vector to the basic feasible solution reduced the number of iterations needed and ultimately cycling could be avoided just because the alternative selection rule did not allow any choice regarding the vector to be taken in or to be dropped from the basis. To upheld this view we would like to discuss the following problem (Balinski and Tucker 1969) in detail.

Problem

Minimize

$$-2x_4 - 3x_5 + x_6 + 12x_7$$

subject to the constraints

$$x_1 - 2x_4 - 9x_5 + x_6 + 9x_7 = 0$$

$$x_2 + \tfrac{1}{3}x_4 + x_5 - \tfrac{1}{3}x_6 - 2x_7 = 0$$

$$x_3 + 2x_4 + 3x_5 - x_6 - 12x_7 = 2$$

and $\quad x_j \geqslant 0, \; j = 1, 2, 3, 4, 5, 6, 7.$

This problem is a cycling *L-P* problem and it completes one cycle at the sixth iteration if normal selection rules for the entering vector are applied. Let us now solve the problem using the alternative selection rules for the entering vector to the basis and represent the results in the following tabular form:

Initial or zero iteration

Basis	c_j	0	0	0	−2	−3	1	12	Solution
		P_1	P_2	P_3	P_4	P_5	P_6	P_7	
P_1	0	1	0	0	−2	−9	1	9	0
P_2	0	0	1	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	−2	0
P_3	0	0	0	1	2	3	−1	−12	1
$z_j - c_j$		0	0	0	2	3	−1	−12	

Here we have $z_4 - c_4 = 2$ and $z_5 - c_5 = 3$.

Corresponding minimum values of θ are $\theta_4 = 0$ and $\theta_5 = 0$. Therefore $\theta_4(z_4 - c_4) = 0$ and $\theta_5(z_5 - c_5) = 0$. So the entering vector** is either P_4 or P_5. Let us choose P_5 as the entering vector then P_2 is to be dropped.

First iteration

P_1	0	1	9	0	1	0	−2	−9	0
P_5	−3	0	1	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	−2	0
P_3	0	0	−3	1	1	0	0	−6	2
$z_j - c_j$		0	−3	0	1	0	0	−6	

Here the vector P_4 is to be taken into the basis and P_1 is to be dropped.

2nd iteration

P_4	−2	1	9	0	1	0	−2	−9	0
P_5	−3	$-\frac{1}{3}$	−2	0	0	1	$\frac{1}{3}$	1	0
P_3	0	−1	−12	1	0	0	2	3	2
$z_j - c_j$		−1	−12	0	0	0	2	3	

In this case $z_6 - c_6 = 2$, $z_7 - c_7 = 3$

$$\theta_6 = 0,\ \theta_7 = 0.\quad \text{So } \theta_6(z_6 - c_6) = 0 = \theta_7\,(z_7 - c_7).$$

So the entering vector is either P_6 or P_7. Let us choose P_7 as the entering vector. P_2 is the vector to be dropped.

3rd iteration

P_4	−2	−2	−9	0	1	9	1	0	0
P_7	12	$-\frac{1}{3}$	−2	0	0	1	$\frac{1}{3}$	1	0
P_3	0	−6	0	1	0	−3	1	0	2
$z_j - c_j$		0	− 6	0	0	−3	1	0	

Here $(z_6 - c_6) = 1$, So P_6 is to be introduced to the basis and P_4 is to be dropped.

**If there are ties, the rule is to select the vector with the lowest or highest index j, vide, Gass (1964, p. 72).

4th iteration

P_6	1	-2	-9	0	1	9	1	0	0
P_7	12	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	-2	0	1	0
P_3	0	2	3	1	-1	-12	0	0	2
$z_j - c_j$		2	3	0	-1	-12	0	0	

$$z_1 - c_1 = 2,\ z_2 - c_2 = 3,\ \theta_1 = 0,\ \theta_2 = 0$$

$$\therefore \quad \theta_1(z_1 - c_1) = 0 = \theta_2(z_2 - c_2).$$

So either P_1 or P_2 is to be taken in. Let us choose P_2 to be the entering vector, then P_7 has to be dropped.

5th iteration

P_6	1	1	0	0	-2	-9	1	9	0
P_2	0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	-2	0	1	0
P_3	0	1	0	1	0	-6	0	-3	2
$z_j - c_j$		1	0	0	0	-6	0	-3	

Here $(z_1 - c_1) = 1$, $\theta_1 = 0$. So P_1 is to be introduced and P_6 is to be dropped.

6th iteration

P_1	0	1	0	0	-2	-9	1	9	0
P_2	0	0	1	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	-2	0
P_3	0	0	0	1	2	3	-1	-12	2
$z_j - c_j$		0	0	0	2	3	-1	-12	

Thus we are back to the state of the initial iteration. In this problem, we find that choices regarding the entering vector to the basis could not be avoided even if alternative selection rules for the entering vector are used. If we could make right choices in iterations 1, 3, and 5, then no cycling is set up and the procedure converges to the optimum.

This can be done with success in all problems using perturbation procedures developed by Dantzig (1953), Charnes (1952) etc.

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