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ON *B*-GRAPHS

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A graph G is a B-graph if every vertex of G is in a maximum independent set. Analogously B^*-graphs are graphs for which each edge is in a maximum independent set of edges. B^{**}-graphs are graphs which are both B-graphs and B^*-graphs. In this paper we obtain some results related to these graphs.

1. INTRODUCTION

We refer to Harary (1969) for the basic definitions and adopt the notation and terminology of Ravindra (1979). A graph is a B-graph if every vertex is in a maximum independent set. If each edge of a graph G is in a maximum independent set of edges, then G is a B^*-graph. B^{**}-graphs are graphs which are B-graphs as well as B^*-graphs. The cardinality of a maximum independent set of edges of a graph is denoted by β. The rth power of a graph G, denoted by G^r, is a graph with $V(G^r) = V(G)$ and $uv \in E(G^r)$ if $d(u, v) \leqslant r$ in G.

2. *B*-GRAPHS

Theorem 2.1 — A path is a B-graph if and only if it is of odd length.

PROOF : First we show that a path of odd length is a B-graph. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of a path of odd length, say P_{2n}. Let $S_1 = \{v_1, v_3, \ldots, v_{2n-1}\}$ and $S_2 = \{v_2, v_4, \ldots, v_{2n}\}$. S_1 and S_2 are independent sets containing n elements each. Hence P_{2n} is a B-graph since $\alpha(P_{2n}) = n$ and $S_1 \cup S_2 = V$.

If a path is a B-graph, then we prove that its length is odd. It is enough to show that a path of even length is not a B-graph. Consider a path P_{2n+1} of even length with vertices $v_1, v_2, \ldots, v_{2n+1}$. Since $\{v_1, v_3, v_5, \ldots, v_{2n+1}\}$ is an independent set, $\alpha(P_{2n+1}) = n + 1$. No vertex with an even suffix is in an independent set containing $n + 1$ elements. Hence P_{2n+1} is not a B-graph.

Theorem 2.2 — Every cycle is a B-graph.

PROOF : Consider an odd cycle first. Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of an odd cycle C_{2n+1}. The sets of vertices $\{v_1, v_3, \ldots, v_{2k-1}, \ldots, v_{2n-1}\}$, $\{v_2, v_4, \ldots, v_{2k}, \ldots v_{2n}\}$ and $\{v_3, v_5, \ldots, v_{2k+1}, \ldots, v_{2n+1}\}$ are independent sets consisting of n elements with V as their union. Since $\alpha(C_{2n+1}) = n$, C_{2n+1} is a B-graph.

Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of an even cycle C_{2n}. The sets $\{v_1, v_3, \ldots, v_{2k-1}, \ldots, v_{2n-1}\}$ and $\{v_2, v_4, \ldots, v_{2k}, \ldots, v_{2n}\}$ are maximum independent sets consisting of n elements each with V as their union. Hence C_{2n} is a B-graph.

Corollary 2.1 — A cycle of even length is a B-point critical graph.

Proof follows by observing that an even cycle C_{2n} is a B-graph and $C_{2n} - v$, for every $v \in V$ is a path of even length which is not a B-graph.

Corollary 2.2 — Odd cycles are B-line critical.

PROOF : An odd cycle C_{2n+1} is a B-graph. $C_{2n+1} - e$, for every $e \in E$ is a path of even length which is not a B-graph.

Theorem 2.3 — An acyclic graph is a B-graph if and only if it has a perfect matching.

PROOF : Suppose that an acyclic graph T has a perfect matching. T is a bipartite graph. Hence $\alpha_0(T) = \beta(T)$ (Konig 1931). Since the matching is perfect, $\beta(T) = p/2$. $\alpha(T) + \alpha_0(T) = p$ (Gallai 1959). Therefore $\alpha(T) = p/2$. So the partition sets V_1 and V_2 of $V(T)$ have cardinality $p/2$ each. Hence T is a B-graph.

Consider an acyclic B-graph G. G is a bipartite graph. Hence $\alpha(G) = p/2$ (Ravindra 1979). Therefore $\alpha_0(G) = p/2$. For bipartite graphs $\alpha_0 = \beta$ (Konig 1931). So $\beta(G) = p/2$. This matching having $p/2$ edges is a perfect matching of G.

Theorem 2.4 — Powers of cycles are B-graphs.

PROOF : Consider the rth $(1 \leqslant r \leqslant n)$ power of an odd cycle C_{2n+1} with vertices $v_1, v_2, \ldots, v_{2n+1}$. Any vertex v_i in C^r_{2n+1} is adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+r}$ and $v_{i+2n}, v_{i+2n-1}, \ldots v_{i+2n-(r-1)}$ (suffix modulo $2n + 1$). Hence a maximum independent set containing v_i is $\{v_i, v_{i+r+1}, v_{i+2r+2}, \ldots, v_{i+lr+l}\}$, where l is the largest integer such that $l < \{2n - (r - 1)\} (r + 1)$. So every vertex is in a maximum independent set consisting of $l + 1$ vertices. Therefore C^r_{2n+1} is a B-graph.

Suppose that C_{2n} is an even cycle with vertices $v_1, v_2, \ldots, v_{2n}$. Every vertex v_i in C^r_{2n} $(1 \leqslant r \leqslant n)$ is adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+r}$ and $v_{i+2n-1}, v_{i+2n-2}, \ldots v_{i+2n-r}$ (suffix modulo $2n$). Hence a maximum independent set containing v_i is

$$\{v_i, v_{i+r+1}, v_{i+2r+2}, \ldots, v_{i+lr+l}\},$$

where l is the largest integer such that $l < (2n - r)(r + 1)^{-1}$. Hence every vertex v_i is in a maximum independent set consisting of $l + 1$ vertices.

Theorem 2.5 — If G_1 and G_2 are B-graphs, then $G_1 + G_2$ is a B-graph if and only if $\alpha(G_1) = \alpha(G_2)$.

PROOF : Let $\alpha(G_1) = \alpha(G_2) = k$. In $G_1 + G_2$ each vertex of G_1 is adjacent to every vertex of G_2. Hence a maximum independent set of $G_1 + G_2$ is either a maximum independent set of G_1 or that of G_2. So $G_1 + G_2$ is a B-graph since

$$V(G_1 + G_2) = V(G_1) \cup V(G_2).$$

Now we will prove that if G_1 and G_2 are B-graphs and $G_1 + G_2$ is a B-graph, then $\alpha(G_1) = \alpha(G_2)$. Suppose that this is not true and let $\alpha(G_1) > \alpha(G_2)$. Then in $G_1 + G_2$ no vertex of G_2 is in a maximum independent set, contradicting the fact that $G_1 + G_2$ is a B-graph.

This theorem gives us a method for generating new B-graphs from known B-graphs. For example P_4, $P_4 + P_4$, $P_4 + P_4 + P_4$, ...; C_n^r, $C_n^r + C_n^r$, $C_n^r + C_n^r + C_n^r$, ...; $C_{2n} + P_{2n}$, $C_{2n} + P_{2n} + P_{2n}$, ... are B-graphs.

Theorem 2.6 — If G_1 and G_2 are B-point critical, then $G_1 + G_2$ is B-point critical if and only if $\alpha(G_1) = \alpha(G_2)$.

PROOF : From Theorem 2.5 we get that if G_1 and G_2 are B-graphs and if $\alpha(G_1) = \alpha(G_2)$, then $G_1 + G_2$ is a B-graph. Since G_1 is B-point critical $G_1 - v$, for every $v \in V(G_1)$ is not a B-graph. That means there is at least one vertex, say v_1 which is not in any maximum independent set of $G_1 - v$. Because v_1 is adjacent to every vertex of G_2, v_1 cannot be in any maximum independent set of $G_1 + G_2 - v$. Similarly for G_2. Hence $G_1 + G_2 - v$, for every $v \in V(G_1 + G_2)$ is not a B-graph.

The second part of the proof is trivial, because, if $\alpha(G_1) \neq \alpha(G_2)$, $G_1 + G_2$ is not even a B-graph.

Theorem 2.6 helps us to construct new B-point critical graphs from known ones. For example C_{2r}, $C_{2r} + C_{2r}$, $C_{2r} + C_{2r} + C_{2r}$, ...;

$$C_{2r}^{r-1}, C_{2r}^{r-1} + C_{2r}^{r-1}, C_{2r}^{r-1} + C_{2r}^{r-1} + C_{2r}^{r-1}, \ldots$$

are all B-point critical graphs.

3. B^*-GRAPHS

Theorem 3.1 — A path is a B^*-graph if and only if it is of even length.

PROOF : Let P_{2n+1} be a path of even length with vertices $v_1, v_2, \ldots, v_{2n+1}$. Let $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, 2n$. The sets $\{e_1, e_3, \ldots, e_{2n-1}\}$ and $\{e_2, e_4, \ldots, e_{2n}\}$ are independent sets having n edges with E as their union. Hence P_{2n+1} is a B^*-graph since $\beta(P_{2n+1}) = n$.

We prove that a path of odd length is not a B^*-graph. Consider a path P_{2n} of odd length with edges $e_1, e_2, \ldots, e_{2n-1}$. The set $\{e_1, e_3, \ldots, e_{2n-1}\}$ is the maximum independent set of edges having n elements. No edge with an even suffix is in an independent set containing n elements. Hence P_{2n} is not a B^*-graph.

Theorem 3.2 — If G is an odd cycle with chords, then G is a B^*-graph.

PROOF : Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of G. Let $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, 2n+1$ (suffix modulo $2n+1$). Let $c = (v_i, v_{i+r})$, $2 \leqslant r \leqslant 2n-1$ be a chord. The sets $S_1 = \{e_1, e_3, \ldots, e_{2n-1}\}$, $S_2 = \{e_2, e_4, \ldots, e_{2n}\}$ and $S_3 = \{e_3, e_5, \ldots, e_{2n+1}\}$ are maximum independent sets since $\beta(G) = n$. Hence each e_i is in a maximum independent set of edges of G. We shall prove that (v_i, v_{i+r}) is in a maximum independent set. If r is odd, say $2k+1$, then $\{(v_{i+1}, v_{i+2}), (v_{i+3}, v_{i+4}), \ldots, (v_{i+2k-1}, v_{i+2k}); (v_i, v_{i+2k+1}); (v_{i+2k+2}, v_{i+2k+3}), (v_{i+2k+4}, v_{i+2k+5}), \ldots, (v_{i+2n-2}, v_{i+2n-1})\}$ is an independent set consisting of n edges. If r is even, say $2k$, then $\{(v_{i+1}, v_{i+2}), (v_{i+3}, v_{i+4}), \ldots, (v_{i+2k-3}, v_{i+2k-2}); (v_i, v_{i+2k}); (v_{i+2k+1}, v_{i+2k+2}), (v_{i+2k+3}, v_{i+2k+4}), \ldots, (v_{i+2n-1}, v_{i+2n})\}$ is an independent set having n edges and containing c. Hence G is a B^*-graph.

Corollary 3.1 — An odd cycle is a B^*-graph.

The proof follows from the fact that each e_i is in a maximum independent set irrespective of the occurrence of any chord c.

Theorem 3.3 — An even cycle with chords which do not make odd cycles is a B^*-graph.

PROOF : Let G be an even cycle with chords satisfying the condition of the Theorem. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of G. Let $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, 2n$ (suffix modulo $2n$). Let $c = (v_i, v_{i+r})$, (suffix modulo $2n$, $2 \leqslant r \leqslant 2n-2$) be a chord which does not make odd cycles. Then r is odd, say $2k+1$. The sets $S_1 = \{e_1, e_3, \ldots, e_{2n-1}\}$ and $S_2 = \{e_2, e_4, \ldots, e_{2n}\}$ have n elements each. Hence each e_i is in a maximum independent set. The edge c is in the maximum independent set $\{(v_{i+1}, v_{i+2}), (v_{i+3}, v_{i+4}), \ldots, (v_{i+2k-1}, v_{i+2k}); (v_i, v_{i+2k+1}); (v_{i+2k+2}, v_{i+2k+3}), (v_{i+2k+4}, v_{i+2k+5}), \ldots, (v_{i+2n-2}, v_{i+2n-1})\}$. Hence G is a B^*-graph.

Corollary 3.2 — An even cycle is a B^*-graph.

Lemma 3.1 — Suppose that G is an even cycle with vertices $v_1, v_2, \ldots, v_{2n}$ and with chords (v_i, v_{i+r}) and (v_{i+1}, v_{i+r+1}), $2 \leqslant r \leqslant 2n-2$ (suffix modulo $2n$), making odd cycles then G is a B^*-graph.

PROOF : Let $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, 2n$. Since (v_i, v_{i+r}) makes odd cycles, r is even say $2k$. We shall prove that the two chords (v_i, v_{i+2k}) and (v_{i+1}, v_{i+2k+1}) form a maximum independent set with $(n-2)$ e_i's. Consider the independent set $\{(v_i, v_{i+2k}), (v_{i+1}, v_{i+2k+1}); (v_{i+2}, v_{i+3}), (v_{i+4}, v_{i+5}), \ldots, (v_{i+2k-2}, v_{i+2k-1}); (v_{i+2k+2}, v_{i+2k+3}), (v_{i+2k+4}, v_{i+2k+5}), \ldots, (v_{i+2n-2}, v_{i+2n-1})\}$. This has n edges. The sets $S_1 = \{e_1, e_3, \ldots, e_{2n-1}\}$ and

$S_2 = \{e_2, e_4, \ldots, e_{2n}\}$ are independent sets having n elements each. Hence every edge of G is in a maximum independent set.

Remark : Since in the proof of the Lemma we could establish that any two chords (v_i, v_{i+r}) and (v_{i+1}, v_{i+r+1}), making odd cycles form a maximum independent set with $(n-2)$ e_i's, it is not necessary that all such chords should be present in G, for G to be a B^*-graph. The Lemma is true even if G is an even cycle with exactly two chords (v_i, v_{i+r}) and (v_{i+1}, v_{i+r+1}) making odd cycles.

Theorem 3.4 — Powers of cycles are B^*-graphs.

PROOF : From Theorem 3.2 we get powers of odd cycles to be B^*-graphs. Any chord in the power of an even cycle either makes odd cycles or does not make odd cycles. In the rth power of an even cycle, the chords (v_i, v_{i+r}) and (v_{i+1}, v_{i+r+1}) are present. Hence by Theorem 3.3 and Lemma 3.1 we get powers of even cycles to be B^*-graphs.

Lemma 3.2 — If an acyclic graph has a perfect matching, then it is unique.

PROOF : We observe that if a graph G has a perfect matching it is of even order. Suppose that an acyclic graph G has two different perfect matchings M_1 and M_2. Let M_1 be $\{(u_1, v_1), (u_2, v_2), \ldots, (u_{p/2}, v_{p/2})\}$. If M_2 is different from M_1, then at least one edge of M_2 is of the form (u_i, v_j), $i \neq j$. Hence the edges (v_i, u_i), (u_i, v_j) and (v_j, u_j) form a P_4 in $M_1 \cup M_2$. Since the components of the subgraph produced by the union of two perfect matchings are k_2's and even length cycles, we conclude that all the components of $M_1 \cup M_2$ are k_2's. Therefore P_4 being a subgraph of $M_1 \cup M_2$, leads to a contradiction.

The result namely "a tree has at most one perfect matching", has been stated as a problem in Behzad *et al.* (1979) and Bondy and Murthy (1976).

Lemma 3.3 — If an acyclic graph has a perfect matching it cannot be a B^*-graph.

PROOF : Let M be the unique perfect matching of an acyclic graph G. Then M contains $p/2$ edges. Since the matching M is unique, there is no perfect matching involving the remaining $\frac{1}{2}p - 1$ edges. Hence these $\frac{1}{2}p - 1$ edges are not in any maximum independent set. Therefore G is not a B^*-graph.

4. B^{**}-GRAPHS

From Theorems 2.1 and 3.1 we see that a path of odd length is a B-graph, but it is not a B^*-graph. A path of even length is a B^*-graph, but it is not a B^{**}-graph. Hence we conclude that a path cannot be a B^{**}-graph. We have a more general result in the following theorem.

Theorem 4.1 — An acyclic graph cannot be a B^{**}-graph.

PROOF : From Theorem 2.3 we get that an acyclic graph is a B-graph if and only if it has a perfect matching. Lemma 3.3 states that if an acyclic graph has perfect matching it cannot be a B^*-graph. Combining these two we see that no acyclic graph is a B^{**}-graph.

Theorem 4.2 — Cycles and their powers are B^{**}-graphs.

PROOF : Combining Theorem 2.2 and Corollaries 3.1 and 3.2 we get that cycles are B^{**}-graphs. From Theorems 2.4 and 3.4 we have powers of cycles to be B^{**}-graphs.

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