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ON FOURIER KERNELS AND ASYMPTOTIC EXPANSION

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The asymptotic expansion and proper conditions of validity for new general Fourier kernels have been investigated.

1. THE A-FUNCTION

In an earlier paper, we (Gautam and Goyal 1980) have introduced a new general transcendental function in the following form:

$$A^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod\limits_1^m \Gamma(a_j-\alpha_j s)\prod\limits_1^n \Gamma(1-b_j+\beta_j s)}{\prod\limits_{m+1}^p \Gamma(1-a_j+\alpha_j s)\prod\limits_{n+1}^q \Gamma(b_j-\beta_j s)}\, x^{-s}\, ds \quad \ldots(1{\cdot}1)$$

where (a) an empty product (i.e. $\prod\limits_j^m$, $m < j$) is interpreted as unity;

(b) m, n, p, q are nonnegative integers with $m \leqslant p$, $n \leqslant q$;

(c) $x(\neq 0)$ and the parameters $a_j, \alpha_j, b_j, \beta_j$ are all complex;

(d) $((a_p, \alpha_p)) \equiv (a_1, \alpha_1), \ldots, (a_p, \alpha_p)$;

(e) $i = \sqrt{-1}$ and the path of integration L is a straight line parallel to the imaginary axis with loops, if necessary, to ensure that the poles of $\prod\limits_1^m \Gamma(a_j - \alpha_j s)$ lie to its right and the poles of $\prod\limits_1^n \Gamma(1 - b_j + \beta_j s)$ lie to its left.

The integral on the right of (1.1) represents an analytic function of x in at least one of the following cases:

(i) $\xi = 0, \eta > 0, |\arg(\zeta x)| < \frac{1}{2}\pi\eta$

(ii) $\xi = 0 = \eta, \nu - \sigma\lambda < -1, x > 0$

where $\xi = \sum_1^p I(\alpha_j) - \sum_1^q I(\beta_j), \; \zeta = \prod_1^p \alpha_j^{\alpha_j} \prod_1^q \beta_j^{-\beta_j},$

$$\eta = \begin{cases} \sum_1^m R(\alpha_j) - \sum_{m+1}^p R(\alpha_j) + \sum_1^n R(\beta_j) - \sum_{n+1}^q R(\beta_j), \\ \text{or} \\ \sum_1^p R(\alpha_j) - \sum_1^q R(\beta_j) - 2 \sum_{m+1}^p |R(\alpha_j)| + 2 \sum_1^n |R(\beta_j)|, \end{cases}$$

$$\nu = \sum_1^p R(a_j) - \sum_1^q R(b_j) - \tfrac{1}{2}(p - q),$$

$$\lambda = \sum_1^p R(\alpha_j) - \sum_1^q R(\beta_j),$$

and $s = \sigma + it$ on the path L when $|t| \to \infty$.

The problems involving unsymmetric Fourier kernels, symmetrical Fourier kernel and asymptotic expansion in connection with the A-function of (1.1) have been discussed in sections 2, 3, 4 respectively. Where as in section 5, we have verified the results obtained in earlier sections.

2. A Pair of A-functions as Unsymmetrical Fourier Kernels

The functions $h(x)$ and $k(x)$ forming a pair of unsymmetrical Fourier kernels are formally expected to be expressible in the following forms (Titchmarsh 1937, §8.3):

$$\left.\begin{aligned} h(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} H(s) x^{-s}\, ds \\ k(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) x^{-s}\, ds \end{aligned}\right\} \quad \ldots(2.1)$$

where $H(s)$ and $K(s)$ [the Mellin-transforms of $h(x)$ and $k(x)$ respectively] satisfy the following functional equation:

$$H(s)\, K(1 - s) = 1. \quad \ldots(2.2)$$

The most general two A-functions which are of the forms (2.1) with their Mellin-transforms satisfying (2.2) are found to be:

$$A^{m,n}_{m+p,n+q}\left[x \,\middle|\, \begin{matrix} ((a_m, \alpha_m)), ((c_p + \gamma_p, \gamma_p)) \\ ((b_n + \beta_n, \beta_n)), ((d_q, \delta_q)) \end{matrix} \right] \equiv A(x)$$

(*equation continued on p.* 1096)

$$= \frac{1}{2\pi i} \int_C \phi(s)\, x^{-s}\, ds,$$

$$\phi(s) = \frac{\prod_1^m \Gamma(a_j - \alpha_j s) \prod_1^n \Gamma(1 - b_j - \beta_j + \beta_j s)}{\prod_1^p \Gamma(1 - c_j - \gamma_j + \gamma_j s) \prod_1^q \Gamma(d_j - \delta_j s)} \qquad \ldots(2.3)$$

and

$$A^{p,q}_{p+m,q+n}\left[x \,\middle|\, \begin{matrix} ((1 - c_p, \gamma_p)), ((1 - a_m + \alpha_m, \alpha_m)) \\ ((1 - d_q + \delta_q, \delta_q)), ((1 - b_n, \beta_n)) \end{matrix} \right] \equiv A'(x)$$

$$= \frac{1}{2\pi i} \int_C \phi'(s)\, x^{-s}\, ds,$$

$$\phi'(s) = \frac{\prod_1^p \Gamma(1 - c_j - \gamma_j s) \prod_1^q \Gamma(d_j - \delta_j + \delta_j s)}{\prod_1^m \Gamma(a_j - \alpha_j + \alpha_j s) \prod_1^n \Gamma(1 - b_j - \beta_j s)} \qquad \ldots(2.4)$$

where the common path of integration C is a straight line drawn parallel to the imaginary axis in such a way that the poles of $\prod_1^m \Gamma(a_j - \alpha_j s)$ and $\prod_1^p \Gamma(1 - c_j - \gamma_j s)$ lie to its right and the poles of $\prod_1^n \Gamma(1 - b_j - \beta_j + \beta_j s)$ and $\prod_1^q \Gamma(d_j - \delta_j + \delta_j s)$ lie to its left.

The functions $A(x)$ and $A'(x)$ will form a pair of unsymmetrical Fourier kernels (Titchmarsh 1937, §8.3) if the following reciprocal equations are satisfied:

$$f(x) = \int_0^\infty A(xy)\, g(y)\, dy, \quad g(x) = \int_0^\infty A'(xy) f(y)\, dy. \qquad \ldots(2.5)$$

In case $A(x) = A'(x)$ the kernel is called symmetric.

On making use of the asymptotic expansion of Γ-function (Whittaker and Watson 1927, §13.6) in the integrands of (2.3) and (2.4), when $s = \sigma + it$ and $|s| = |t| \to \infty$ on C, we obtain

$$\phi(s)x^{-s} = |t|^{\sigma(P+Q)-Q+M} \exp[it\{(P + Q)(\log|t| - 1) - \log(xw) + iQ \arg s + iP \arg(-s)\}][A + O(|t|^{-1})] \qquad \ldots(2.6)$$

and

$$\phi'(s)x^{-s} = -\mid t \mid^{\sigma(P+Q)-P-M} \exp[it\{(P+Q)(\log \mid t \mid - 1) - \log(xw) + iP \arg s + iQ \arg(-s)\}][A' + O(\mid t \mid^{-1})] \quad \ldots(2.7)$$

where $P = -\sum_1^m \alpha_j + \sum_1^q \delta_j,\ Q = \sum_1^n \beta_j - \sum_1^p \gamma_j,$

$$M = \sum_1^m (a_j - \tfrac{1}{2}) - \sum_1^n (b_j - \tfrac{1}{2}) + \sum_1^p (c_j - \tfrac{1}{2}) - \sum_1^q (d_j - \tfrac{1}{2}),$$

$$w = \prod_1^m \alpha_j^{\alpha_j} \prod_1^n \beta_j^{-\beta_j} \prod_1^p \gamma_j^{\gamma_j} \prod_1^q \delta_j^{-\delta_j},$$

A and A' are both constants, each of which may have one value for large positive t and another value for large negative t.

If the various parameters are so chosen that $R(P) = \frac{1}{2}D = R(Q), I(P) = 0 = I(Q)$, $M = 0$ then from (2.6) and (2.7) we have

$$\phi(s)x^{-s} = \phi'(s)\,x^{-s} = \mid t \mid^{(2\sigma-1)D/2} \exp[it\{D(\log \mid t \mid - 1) - \log(xw)\}] \times [A(\text{or } A') + O(\mid t \mid^{-1})].$$

This gives us the behaviour of the integrands of (2.3) and (2.4). Now working on the analysis* similar to that of Kesarwani (1965, §3, 4), by making essential algebraical changes, we have obtained the following two theorems each of which establishes the reciprocal equations (2.5).

Theorem 1 — If (i) $A_1(x) = \int_0^x A(u)\,du,\ A_1'(x) = \int_0^x A'(u)\,du,$

(ii) $\left[-\sum_1^m R(\alpha_j) + \sum_1^q R(\delta_j)\right] = \frac{D}{2} = \left[\sum_1^n R(\beta_j) - \sum_1^p R(\gamma_j)\right] > 0,$

$$\sum_1^m I(\alpha_j) = \sum_1^q I(\delta_j),\ \sum_1^n I(\beta_j) = \sum_1^p I(\gamma_j),$$

$$\sum_1^m (a_j - \tfrac{1}{2}) - \sum_1^n (b_j - \tfrac{1}{2}) + \sum_1^p (c_j - \tfrac{1}{2}) - \sum_1^q (d_j - \tfrac{1}{2}) = 0,$$

(iii) $\min R\left(\frac{a_j + \mu}{\alpha_j}\right) > \frac{1}{2},\ \alpha_j \neq 0\ (j = 1, \ldots, m);$

$\max R\left(\frac{1 - b_j + \nu}{\beta_j}\right) > \frac{1}{2},\ \beta_j \neq 0\ (j = 1, \ldots, n);$

*Originated by Fox (1961, §6, 7, 8).

$$\min R\left(\frac{1-c_j+\mu'}{\gamma_j}\right) > \tfrac{1}{2}, \gamma_j \neq 0 \; (j = 1, \ldots, p);$$

$$\max R\left(\frac{d_j+\nu'}{\delta_j}\right) > \tfrac{1}{2}, \delta_j \neq 0 \; (j = 1, \ldots, q);$$

where $\mu, \mu', \nu, \nu' = 0, 1, 2, \ldots.$

(iv) $f(x) \in L^2(0, \infty)$, then

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{A_1(xu)}{u} f(u)\, du$$

and

$$g'(x) = \frac{d}{dx}\int_0^\infty \frac{A'_1(xu)}{u} f(u)\, du$$

define almost everywhere in $(0, \infty)$ the functions g and g' respectively, both belonging to $L^2(0, \infty)$, also

$$\frac{d}{dx}\int_0^\infty \frac{A_1(xu)}{u} g'(u)\, du = f(x) = \frac{d}{dx}\int_0^\infty \frac{A'_1(xu)}{u} g(u)\, du$$

holds almost everywhere in $(0, \infty)$ and

$$\int_0^\infty [f(x)]^2\, dx = \int_0^\infty g(x)\, g'(x)\, dx.$$

Theorem 2 — If the condition (ii) of Theorem 1 holds and

(I) $$\min R\left(\frac{a_j+\mu}{\alpha_j}\right) > \frac{D+1}{2D}, \alpha_j \neq 0 \; (j = 1, \ldots, m);$$

$$\max R\left(\frac{1-b_j+\nu}{\beta_j}\right) \geqslant \frac{D+1}{2D}, \beta_j \neq 0 \; (j = 1, \ldots, n);$$

$$\min R\left(\frac{1-c_j+\mu'}{\gamma_j}\right) > \frac{D+1}{2D}, \gamma_j \neq 0 \; (j = 1, \ldots, p);$$

$$\max R\left(\frac{d_j+\nu'}{\delta_j}\right) \geqslant \frac{D+1}{2D}, \delta_j \neq 0 \; (j = 1, \ldots, q);$$

where $\mu, \mu', \nu, \nu' = 0, 1, 2, \ldots,$

(II) $f(y)\, y^{(1-D)/2D} \in L(0, \infty)$ and $f(y)$ is of bounded variation near $y = x$, $x > 0$ then

$$\int_0^\infty A(xu) \left[\int_0^\infty A'(yu) f(y)\, dy\right] du = \tfrac{1}{2}[f(x+) + f(x-)].$$

3. Deducing an A-function as a Symmetrical Fourier Kernel

In order to make $A(x)$ and $A'(x)$, of respectively (2.3) and (2.4), to be identical we suppose $p = m$, $q = n$, $1 - c_j = a_j$, $\gamma_j = \alpha_j$, $1 - b_j = d_j$, $\beta_j = \delta_j$. Therefore

$$A(x) = A'(x) = A^{m,n}_{2m,2n}\left[x \,\middle|\, \begin{matrix} ((a_m, \alpha_m)), ((1 - a_m + \alpha_m, \alpha_m)) \\ ((1 - d_n + \delta_n, \delta_n)), ((d_n, \delta_n)) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_1^m \Gamma(a_j - \alpha_j s) \prod_1^n \Gamma(d_j - \delta_j + \delta_j s)}{\prod_1^m \Gamma(a_j - \alpha_j + \alpha_j s) \prod_1^n \Gamma(d_j - \delta_j s)} x^{-s}\, ds \qquad \ldots(3.1)$$

is a symmetrical Fourier kernel provided

$$\tfrac{1}{2} D = \{- \sum_1^m R(\alpha_j) + \sum_1^n R(\delta_j)\} > 0,\ \sum_1^m I(\alpha_j) = \sum_1^n I(\delta_j).$$

4. Asymptotic Expansion

The asymptotic expansion of the first kernel $A(x)$, of (2.3), for large and real positive x with an error term $O(x^{-\delta})$, for a given δ, is contained in the following theorem.

Theorem 3 — If the function $A(x)$, of (2.3), is such that

(i) $$[- \sum_1^m R(\alpha_j) + \sum_1^q R(\delta_j)] = \tfrac{1}{2} D = [\sum_1^n R(\beta_j) - \sum_1^p R(\gamma_j)] > 0,$$

$$\sum_1^m I(\alpha_j) = \sum_1^q I(\delta_j),\ \sum_1^n I(\beta_j) = \sum_1^p I(\gamma_j),$$

$$\sum_1^m (a_j - \tfrac{1}{2}) - \sum_1^n (b_j - \tfrac{1}{2}) + \sum_1^p (c_j - \tfrac{1}{2}) - \sum_1^q (d_j - \tfrac{1}{2}) = 0,$$

$$\tfrac{1}{2} K = \sum_1^q d_j - \sum_1^m a_j,\ \alpha = D^{-D} \prod_1^m \alpha_j^{-\alpha_j} \prod_1^n \beta_j^{\beta_j} \prod_1^p \gamma_j^{-\gamma_j} \prod_1^q \delta_j^{\delta_j},$$

(ii) $$\min R\left(\frac{a_j + u}{\alpha_j}\right) > \tfrac{1}{2},\ \alpha_j \neq 0\ (j = 1, \ldots, m);$$

$$\max R\left(\frac{1 - b_j + v}{\beta_j}\right) > 1,\ \beta_j \neq 0\ (j = 1, \ldots, n);$$

$u, v = 0, 1, 2, \ldots,$

(iii) x is real positive and $R(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_q) > 0$,

(iv) given $\delta > \max R\left(\frac{a_1}{\alpha_1}, \ldots, \frac{a_m}{\alpha_m}\right)$, r denotes the greatest integer in

$$D(\delta - \tfrac{1}{2}) + \tfrac{3}{2} \text{ and } u_h(\forall h = 1, \ldots, m)$$

denotes the greatest nonnegative integer less than

$$\{\delta \mid \alpha_h \mid^2 - R(a_h)\, R(\alpha_h) - I(a_h)\, I(\alpha_h)\}/R(\alpha_h)$$

then

$$A(x) = \sum_{i=0}^{r} \frac{v_i}{D} \left(\frac{x}{\alpha}\right)^{z/D} \begin{matrix}\sin\\ \cos\end{matrix} \left\{\frac{\pi}{2}(K+z) - \left(\frac{x}{\alpha}\right)^{1/D}\right\}$$

$$+ \sum_{h=1}^{m} \sum_{\nu=0}^{u_h} \frac{(-)^\nu x^{-w}}{\nu!\, \alpha_h} \phi(w) + O(x^{-\delta}) \qquad \ldots(4.1)$$

where the expression on the right of (4.1) be read with sin or cos according as $q - m$ is an odd or even positive integer,

$$z = \frac{1-D}{2} - i, \quad w = \frac{a_h + \nu}{\alpha_h},$$

$$\phi(w) = \frac{\prod_{j=1, j\neq h}^{m} \Gamma(a_j - \alpha_j w) \prod_{j=1}^{n} \Gamma(1 - b_j - \beta_j + \beta_j w)}{\prod_{j=1}^{p} \Gamma(1 - c_j - \gamma_j + \gamma_j w) \prod_{j=1}^{q} \Gamma(d_j - \delta_j w)}$$

and $v_0, v_1, \ldots, v_r$ are constants depending on $a_j, \alpha_j, b_j, \beta_j, c_j, \gamma_j, d_j, \delta_j$.

PROOF : Rewriting (2.3) as

$$A(x) = \frac{1}{2\pi i} \int_C Q(s) \left(\frac{x}{\alpha}\right)^{-s} ds, \; Q(s) = \phi(s)\, \alpha^{-s}. \qquad \ldots(4.2)$$

In view of condition (ii) of the statement the contour C of (4.2) can be taken as $\sigma = \mu$, where $0 < \mu < \frac{1}{2}$.

The use of $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}(\pi z)$ gives,

$$Q(s) = (\text{g.f.})\,(\text{s.f.})\, \pi^{m-q} \alpha^{-s} \qquad \ldots(4.3)$$

where

$$(\text{g.f.}) = \frac{\prod_1^n \Gamma(1 - b_j - \beta_j + \beta_j s) \prod_1^q \Gamma(1 - d_j + \delta_j s)}{\prod_1^p \Gamma(1 - c_j - \gamma_j + \gamma_j s) \prod_1^m \Gamma(1 - a_j + \alpha_j s)},$$

$$(\text{s.f.}) = \prod_1^q [\sin\{\pi(d_j - \delta_j s)\}] \prod_1^m [\operatorname{cosec}\{\pi(a_j - \alpha_j s)\}]. \qquad \ldots(4.4)$$

On using the asymptotic expansion of gamma-function in the form due to Fox (1961, p. 418), we obtain

$$(\text{g.f.}) = F \exp\{M \log s + N + s \log(\alpha D^D) + s(P+Q)(\log s - 1) - Q \log s\} \quad \ldots(4.5)$$

where the function F is as defined by Fox in his aforesaid paper, N is a constant independent of s, indeed,

$$N = \sum_1^m (a_j - \tfrac{1}{2}) \log \alpha_j - \sum_1^n (b_j - \tfrac{1}{2}) \log \beta_j - \sum_1^n \beta_j \log \beta_j + \sum_1^p (c_j - \tfrac{1}{2}) \log \gamma_j - \sum_1^q (d_j - \tfrac{1}{2}) \log \delta_j + \sum_1^p \gamma_j \log \gamma_j$$

and M, P, Q are as in section 2.

The use of condition (i) of the statement in (4.5) gives

$$(\text{g.f.}) = F \exp\{N + s \log \alpha + sD(\log sD - 1) - \tfrac{1}{2} D \log s\}$$

which after a little simplification and absorption of the constant term $N + \frac{1}{2} \log D$ in F gives

$$(\text{g.f.}) = F s^{(1-D)/2} \alpha^s \Gamma(Ds) \quad \ldots(4.6)$$

Let $s = \sigma + it$, where σ is fixed and t is large positive or negative. Then it is easy to see that

$$\sin \pi(d_j - \delta_j s) = \begin{cases} \dfrac{1}{2i}[\exp\{i\pi(d_j - \delta_j s)\}][1 + AI \exp\{-2\pi t\, R(\delta_j)\}] \\ -\dfrac{1}{2i}[\exp\{-i\pi(d_j - \delta_j s)\}]\left[1 + \dfrac{1}{AI} \exp\{2\pi t R(\delta_j)\}\right] \end{cases} \quad \ldots(4.7)$$

where A is a constant independent of t and I is a purely imaginary quantity. Suppose

$$c = 2\pi \min R(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_q).$$

Then from the condition (iii) of the statement

$$0 < c < 2\pi R(\delta_j).$$

Also t is equal to $|t|$ or $-|t|$ according as $t >$ or < 0. Therefore from (4.7) we have when $|t| \to \infty$,

$$\sin \pi(d_j - \delta_j s) = \begin{cases} \dfrac{1}{2i}[\exp\{i\pi(d_j - \delta_j s)\}][1 + O(e^{-c|t|})], \ t > 0. \\ -\dfrac{1}{2i}[\exp\{-i\pi(d_j - \delta_j s)\}][1 + O(e^{-c|t|})], \ t < 0. \end{cases}$$

The use of this result and a similar other result in (4.4) gives, after some simplification,

$$(\text{s.f.}) = \begin{cases} (2i)^{m-q}[\exp\{\tfrac{1}{2}i\pi(K-Ds)\}][1+O(e^{-c|t|})], & t>0 \\ (-2i)^{m-q}[\exp\{-\tfrac{1}{2}i\pi(K-Ds)\}][1+O(e^{-c|t|})], & t<0. \end{cases} \qquad \ldots(4.8)$$

But $(2i)^{m-q}$ is purely imaginary or purely real according as $q-m$ is odd or even. Therefore, whether $t >$ or < 0, each of the expressions of (4.8) reduces to

$$\begin{cases} \lambda_1[\sin\{\tfrac{1}{2}\pi(K-Ds)\}][1+O(e^{-c|t|})], & q-m \text{ is odd} \\ \lambda_2[\cos\{\tfrac{1}{2}\pi(K-Ds)\}][1+O(e^{-c|t|})], & q-m \text{ is even} \end{cases} \qquad \ldots(4.9)$$

where λ_1 and λ_2 are real numbers given by

$$\lambda_1 = (2i)^{m-q+1} \text{ and } \lambda_2 = 2(2i)^{m-q}.$$

Thus from (4.9) we obtain

$$(\text{s.f.}) = \begin{cases} [\sin\{\tfrac{1}{2}\pi(K-Ds)\}][\lambda_1+O(e^{-c|t|})], & q-m \text{ is odd,} \\ [\cos\{\tfrac{1}{2}\pi(K-Ds)\}][\lambda_2+O(e^{-c|t|})], & q-m \text{ is even.} \end{cases} \qquad \ldots(4.10)$$

The use of (4.6) and (4.10) in (4.3) gives

$$Q(s) = Fs^{(1-D)/2}\,\Gamma(Ds)\,\pi^{m-q} \times \begin{cases} [\sin\{\tfrac{1}{2}\pi(K-Ds)\}][\lambda_1+O(e^{-c|t|})], & q-m \text{ is odd} \\ [\cos\{\tfrac{1}{2}\pi(K-Ds)\}][\lambda_2+O(e^{-c|t|})], & q-m \text{ is even.} \end{cases}$$

Since $\pi^{m-q}[(\lambda_1 \text{ or } \lambda_2) + O\, e^{-c|t|})]$ can be amalgamated into F, therefore

$$Q(s) = Fs^{(1-D)/2}\,\Gamma(Ds)\,\begin{matrix}\sin\\ \cos\end{matrix}\{\tfrac{1}{2}\pi(K-Ds)\} \qquad \ldots(4.11)$$

where on the right read sin or cos according as $q-m$ is odd or even.

Following the discussion due to Fox (1961), the establishment of (4.1) requires the evaluation of the following integral:

$$I = \frac{1}{2\pi i}\int_{\sigma=\delta} R(s)\left(\frac{x}{\alpha}\right)^{-s} ds \qquad \ldots(4.12)$$

where

$$R(s) = Q(s) - \sum_{i=0}^{r} \nu_i \Gamma\left(Ds - i + \frac{1-D}{2}\right)\begin{matrix}\sin\\ \cos\end{matrix}\{\tfrac{1}{2}\pi(K-Ds)\}.$$

The straight line path $\sigma = \delta$ of I is such that at least one pole of $\prod_1^m \Gamma(a_j - \alpha_j s)$ lies to its left. This is possible if δ is greater than the maximum of the real parts of the poles $(a_j + \nu)/\alpha_j$ ($j = 1, \ldots, m$) for $\nu = 0$, which is covered by the (iv) condition of the statement. To evaluate I we divide the residues to the left of $\sigma = \delta$ into three groups.

Group 1 contains all the residues of $\dfrac{1}{2\pi i} Q(s) \left(\dfrac{x}{\alpha}\right)^{-s}$ which arise from the poles of $\prod_1^n \Gamma(1 - b_j - \beta_j + \beta_j s)$. The condition (ii) of the statement ensures that all these poles lie to the left of $\sigma = 0$ and therefore certainly to the left of $\sigma = \delta$ also. Evidently these residues are the same as those of the integral in (4.2), which defines $A(x)$. Hence the sum of the residues in this group is $A(x)$.

Group 2 contains all the residues of $\dfrac{1}{2\pi i} Q(s) \left(\dfrac{x}{\alpha}\right)^{-s}$ which arise from those poles of $\prod_1^m \Gamma(a_j - \alpha_j s)$, lying to the left of $\sigma = \delta$, According to condition (ii) these poles lie to the right of $\sigma = \frac{1}{2}$, so that all the poles of this group lie between the lines $\sigma = \frac{1}{2}$ and $\sigma = \delta$. All the poles of $\prod_1^m \Gamma(a_j - \alpha_j s)$ are given by

$$\left\{\frac{a_h + u}{\alpha_h} : u = 0, 1, 2, \ldots\right\} \quad (\forall h = 1, \ldots, m). \qquad \ldots(4.13)$$

Suppose the poles (4.13) for $u = 0, 1, 2, \ldots, u_h$ lie to the left of $\sigma = \delta$. That is u_h is the greatest non-negative integer such that

$$R\,((a_h + u_h)/\alpha_h) < \delta < R\,((a_h + u_h + 1)/\alpha_h). \qquad \ldots(4.14)$$

If $a_h = a + ib$ and $\alpha_h = c + id$ then the first part of (4.14) can be rewritten as

$$R\left\{\frac{a + ib + u_h}{c + id} \; \frac{c - id}{c - id}\right\} < \delta$$

i.e. $$\frac{(a + u_h)\,c + bd}{|\,\alpha_h\,|^2} < \delta$$

i.e. $$u_h < \{\delta\,|\,\alpha_h\,|^2 - ac - bd\}/c$$

which is covered in (iv) condition of the statement. Thus the poles of this group 2 are as given by (4.13) for $u = 0, 1, 2, \ldots, u_h$ ($\forall h = 1, \ldots, m$). Hence the corresponding residues are

$$-\frac{1}{2\pi i} \sum_{h=1}^{m} \sum_{u=0}^{u_h} \phi\left(\frac{a_h + u}{\alpha_h}\right) \frac{(-)^u\, x^{-(a_h+u)/\alpha_h}}{u!\,\alpha_h}\,.$$

Group 3 contains all the residues of

$$-\frac{1}{2\pi i}\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{\pi}{2}(K-Ds)\right\}\left\{\sum_{i=0}^{r}\nu_i\Gamma\left(Ds-i+\frac{1-D}{2}\right)\right\}\left(\frac{x}{\alpha}\right)^{-s} \quad \ldots(4.15)$$

arising from the poles of this expression which lie to the left of $\sigma=\delta$. Fox (1961) has shown that even if a pole of (4.15) lies to the right of $\sigma=\delta$ still (4.15) can be dealt as if all its poles lie to the left of $\sigma=\delta$. The residues of (4.15) at its poles are

$$=-\frac{1}{2\pi i}\sum_{\nu=0}^{\infty}\sum_{i=0}^{r}\frac{\nu_i(-)^{\nu}}{D\nu!}\left(\frac{x}{\alpha}\right)^{(z+\nu)/D}\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{\pi}{2}(K+z+\nu)\right\},$$

or

$$=-\frac{1}{2\pi i}\sum_{i=0}^{r}\frac{\nu_i}{D}\left(\frac{x}{\alpha}\right)^{z/D}\sum_{\nu=0}^{\infty}\left\{-\left(\frac{x}{\alpha}\right)^{1/D}\right\}^{\nu}$$

$$\times\frac{1}{\nu!}\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{\pi}{2}(K+z+\nu)\right\},$$

or

$$=-\frac{1}{2\pi i}\sum_{i=0}^{r}\frac{\nu_i}{D}\left(\frac{x}{\alpha}\right)^{z/D}\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{\pi}{2}(K+z)-\left(\frac{x}{\alpha}\right)^{1/D}\right\},$$

on using the following trigonometrical result

$$\sum_{r=0}^{\infty}\frac{x^r}{r!}\begin{matrix}\sin\\ \cos\end{matrix}(\alpha+r\beta)=e^{x\cos\beta}\begin{matrix}\sin\\ \cos\end{matrix}(\alpha+x\sin\beta).$$

By Cauchy's theorem the sum of residues in these three groups is equal to I. Fox (1961) has also shown that I is of the order of $x^{-\delta}$. Hence (4.1) is established.

5. Known Special Cases

(i) If α_j, β_j, γ_j, δ_j are taken to be real positive and a_j, b_j, c_j, d_j are replaced respectively by $a_j+\frac{1}{2}\alpha_j$, $1-b_j-\frac{1}{2}\beta_j$, $1-c_j-\frac{1}{2}\gamma_j$, $d_j+\frac{1}{2}\delta_j$ then the functions $A(x)$ and $A'(x)$, of (2.3) and (2.4), reduce to Kesarwani's (1965) unsymmetrical Fourier kernels $H^{(1)}(x)$ and $H^{(2)}(x)$. In this case (4.1) will give the asymptotic expansion of $H^{(1)}(x)$ (Kesarwani 1965, 1972).

(ii) If α_j, δ_j are taken to be real positive and d_j be replaced by $d_j+\delta_j$ in (3.1) then we obtain Fox's (1961) symmetrical Fourier kernel $H(x)$. Then the asymptotic expansion of $H(x)$, due to Fox (1961), can easily be deduced from (4.1).

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Added in the Proof

The *A*-function has been extended to 'two' and *n* variables [see Gautam, G. P. '*A* study of general transcendental functions'—Ph.D. *Thesis*, Rajasthan University, 1981; where an attempt has been made to close, at the present moment, the possibility of further generalization and unify the study in Fourier kernels, asymptotic expansions, integrals and integral equations etc.]. Consequently, the multivariate *H*-functions of R. K. Saxena [*Kyungpook Math. J.*, **14** & **17** (1974, 1977) pp. 255–259 & 221–228] and H. M. Srivastava, and R. Panda [*J. Reine angew. Math.*, **283/284** (1976)] become special cases of *A*-function of multivariate character.