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MATRIX TRANSFORMATIONS OF $c_0(p)$, $l^\infty(p)$, l^p AND l INTO χ

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Necessary and sufficient conditions have been established for an infinite matrix $A = (a_{nk})$ to transform $c_0(p)$, $l^\infty(p)$, l^p and l into χ where $c_0(p)$ and $l^\infty(p)$ are sequence spaces of Maddox (1967) and χ that of Kamthan (1976).

1. INTRODUCTION

Γ, the space of entire sequences introduced by Ganapathy Iyer (1948) and its dual Γ^α become particular cases respectively of $c_0(p)$ and $l^\infty(p)$, the generalized sequence spaces introduced by Maddox (1967). Kamthan (1976) introduced the sequence space χ. If (λ, μ) denote the set of all matrices $A = (a_{nk})$; $n, k = 1, 2, 3, \ldots$, which transforms the sequence space λ into the sequence space μ, Sridhar (1979) obtained necessary and sufficient conditions for $A \in (c_0, \chi)$. In this paper we obtain conditions to characterize $(c_0(p), \chi)$ matrices which generalizes the result of Sridhar (1979). Also conditions have been obtained to characterize $(l^\infty(p), \chi)$, (l^p, χ) and(l, χ) matrices.

In § 2 we deal with definitions and quote some known results as lemmas which will be used in §3 for establishing conditions to characterize $(c_0(p), \chi)$, $(l^\infty(p), \chi)$, (l^p, χ) and (l, χ) matrices.

For unexplained terms in sequence spaces and matrix transformations thereon and for elements on sequence spaces, we refer respectively Kamthan and Gupta (1981) and Köthe (1960).

2. DEFINITIONS AND SOME KNOWN RESULTS

Let ω denote the space of all sequences and e^k the kth unit vector. A subspace λ of ω such that λ contains the space generated by $\{e^k; k \geqslant 1\}$ is called a sequence space. By λ^α we mean Köthe-Toeplitz dual of λ. Suppose μ is another sequence space. Denote by $A = (a_{nk})$ an arbitrary matrix transformation from λ to μ, thus if $y = Ax$, then $y = (y_n) \in \mu$ with $y_n = \sum_{k \geqslant 1} a_{nk} x_k$, $\forall\ x = (x_k)$ in λ.

If $p = (p_k)$ is a sequence where p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$, we

define

$$c_0(p) = \{x = (x_k) : | x_k |^{p_k} \to 0 \text{ as } k \to \infty\}$$

$$l^\infty(p) = \{x = (x_k) : \sup_k | x_k |^{p_k} < \infty\}$$

$$l^p = \{x = (x_k) : \sum_{k \geqslant 1} | x_k |^p < \infty; p > 1\}$$

$$l = \{x = (x_k) : \sum_{k \geqslant 1} | x_k | < \infty\}$$

$$\chi = \{x = (x_k) : (k\,!\, | x_k |)^{1/k} \to 0 \text{ as } k \to \infty\}$$

$$\Gamma = \{x = (x_k) : | x_k |^{1/k} \to 0 \text{ as } k \to \infty\}$$

$$\Gamma^x = \{x = (x_k) : \sup_k | x_k |^{1/k} < \infty\}$$

$$c_0 = \{x = (x_k) : (x_k) \text{ is null}\}$$

$$l^\infty = \{x = (x_k) : (x_k) \text{ is bounded}\}.$$

Remark: χ can be regarded as the collection of all entire functions $f(z) = \sum_{k \geqslant 1} x_k z^k$ of exponential order 1 and type 0.

Now we quote some known results as the following Lemmas :

Lemma A (Maddox 1969) — $c_0^x(p) = \bigcup_{M>1} \{(a_k) : \sum_{k \geqslant 1} | a_k | M^{-1/p_k} < \infty\}$.

Lemma B (Lascarides and Maddox 1970) —

$$(l^\infty(p))^x = \bigcap_{M=2}^{\infty} \{(a_k) : \sum_{k \geqslant 1} | a_k | M^{1/p_k} < \infty\}.$$

Lemma C (Cooke 1955) — $l^x = l^\infty$.

Lemma D (Cooke 1955) — $(l^p)^x = l^q$ where $p > 1$ and $p^{-1} + q^{-1} = 1$.

3. Main Results

Theorem 1 — $A \in (c_0(p), \chi)$ if and only if, for some integer $M > 1$,

$$(n! \sum_{k \geqslant 1} | a_{nk} | M^{-1/p_k})^{1/n} \to 0 \text{ as } n \to \infty \qquad \ldots(1)$$

Proof : *Sufficiency* — Since $(x_k) \in c_0(p)$, there exists an $M > 1$ such that

$$| x_k | \leqslant M^{-1/p_k}$$

for all k. Let (1) hold, then for a given $\epsilon > 0$, there exists a n_0 such that

$$(n\,! \sum_{k \geqslant 1} | a_{nk} | M^{-1/p_k})^{1/n} < \epsilon \text{ for } n \geqslant n_0 \qquad \ldots(2)$$

Now

$$(n! \mid y_n \mid)^{1/n} \leqslant (n! \sum_{k \geqslant 1} \mid a_{nk} \mid \mid x_k \mid)^{1/n} \leqslant (n! \sum_{k \geqslant 1} \mid a_{nk} \mid M^{-1/p_k})^{1/n}$$

$$< \epsilon \text{ for } n \geqslant n_0 \text{ using (2)}$$

Hence $(y_n) \in \chi$.

Necessity — If (1) does not hold, there exists subsequences of (n) such that

$$(n! \sum_{k \geqslant 1} \mid a_{nk} \mid M^{-1/p_k})^{1/n} > \epsilon \text{ when } n \to \infty. \qquad ...(3)$$

Since the matrix (a_{nk}) is applicable to each member of $c_0(p)$, $(a_{nk}) \in c_0^*(p)$ so that by Lemma A,

$$\sum_{k \geqslant 1} \mid a_{nk} \mid M^{-1/p_k} < \infty \text{ for } M > 1. \qquad ...(4)$$

Since $x = e^k \in c_0(p)$, $(y_n) = (a_{nk}) \in \chi$ so that $(n! \mid a_{nk} \mid)^{1/n} \to 0$ as $n \to \infty$ for each fixed k. Hence

$$(n! \mid a_{nk} \mid)^{1/n} \leqslant A_k \text{ for all } n \text{ and for each fixed } k. \qquad ...(5)$$

Now we shall construct a sequence $(x_k) \in c_0(p)$ and show that the corresponding $(y_n) \notin \chi$ using (3), (4) and (5).

Then that will suffice to prove that the condition is necessary.

By (3), we can choose $n = n_1$ and $k = q_1$ such that

$$\left(n_1! \sum_{k=1}^{q_1} \mid a_{n_1 k} \mid (M + 1)^{-1/p_k}\right)^{1/n_1} > 1. \qquad ...(6)$$

Having fixed n_1, by (4) choose $k = k_1 > q_1$ such that

$$\left(n_1! \sum_{k=k_1+1}^{\infty} \mid a_{n_1 k} \mid (M + 1)^{-1/p_k}\right)^{1/n_1} < \epsilon. \qquad ...(7)$$

Taking for all n,

$$\left.\begin{aligned} x_k &= \operatorname{sgn}(a_{nk})\,(M + 1)^{-1/p_k} \text{ for } 1 \leqslant k \leqslant k_1 \\ &= \operatorname{sgn}(a_{nk})\,(M + i)^{-1/p_k} \text{ for } k_{i-1} < k \leqslant k_i;\ i = 2, 3, \ldots \end{aligned}\right\} \qquad ...(8)$$

so that $(x_k) \in c_0(p)$ and

$$(M + i)^{-1/p_k} \leqslant (M + i - 1)^{-1/p_k} \qquad ...(9)$$

we have

$$(n_1!\,|\,y_{n_1}\,|\,)^{1/n_1} \geqslant \left(n_1!\left|\sum_{k=1}^{k_1} a_{n_1k}\,x_k\right|\right)^{1/n_1}$$

$$-\left(n_1!\left|\sum_{k=k_1+1}^{\infty} a_{n_1k}x_k\right|\right)^{1/n_1}$$

$$\geqslant (n_1!\ \sum_{k=1}^{k_1}\ |\,a_{n_1k}\,|\,(M+1)^{-1/p_k})^{1/n_1}$$

$$-\left(n_1!\ \sum_{k=k_1+1}^{\infty}\left|a_{n_1k}\,\right|(M+2)^{-1/p_k}\right)^{1/n_1}$$

[using (8)]

$$> 1 - \epsilon \text{ [using (6), (9) and (7)].}$$

From (9) and (5) we have for all n,

$$\left(n!\sum_{k=1}^{k_i}\,|\,a_{nk}\,|\,(M+i)^{-1/p_k}\right)^{1/n} \leqslant \left(n!\sum_{k=1}^{k_i}\,|\,a_{nk}\,|\,M^{-1/p_k}\right)^{1/n}$$

$$\leqslant C_{k_i} \text{ where } C_{k_i} = \sum_{k=1}^{k_i} A_k. \quad \ldots(10)$$

By (3) choose $n = n_2 > n_1$ and $q_2 > k_1$ such that

$$\left(n_2!\ \sum_{k=k_1+1}^{q_2}\ |\,a_{n_2k}\,|\,(M+2)^{-1/p_k}\right)^{1/n_2} > 2 + C_{k_1}. \quad \ldots(11)$$

Having fixed n_2, by (4) choose $k = k_2 > q_2$ such that

$$\left(n_2!\ \sum_{k=k_2+1}^{\infty}\ |\,a_{n_2k}\,|\,(M+2)^{-1/p_k}\right)^{1/n_2} < \epsilon. \quad \ldots(12)$$

Hence

$$(n_2!\,|\,y_{n_2}\,|\,)^{1/n_2} \leqslant \left(n_2!\left|\sum_{k=k_1+1}^{k_2} a_{n_2k}x_k\right|\right)^{1/n_2} - \left(n_2!\left|\sum_{k=1}^{k_1} a_{n_2k}x_k\right|\right)^{1/n_2}$$

$$-\left(n_2!\left|\sum_{k=k_2+1}^{\infty} a_{n_2k}x_k\right|\right)^{1/n_2}$$

(*equation continued on p.* 1110)

$$\geqslant \left(n_2! \sum_{k=k_1+1}^{k_2} |a_{n_2k}| (M+2)^{-1/p_k}\right)^{1/n_2}$$

$$- \left(n_2! \sum_{k=1}^{k_1} |a_{n_2k}| (M+1)^{-1/p_k}\right)^{1/n_2}$$

$$- \left(n_2! \sum_{k=k_2+1}^{\infty} |a_{n_2k}| (M+3)^{-1/p_k}\right)^{1/n_2} \text{[using (8)]}$$

$$> 2 - \epsilon \text{ [using (11), (10), (9) and (12)].}$$

Proceeding like this, by (3) we can choose $n_i > n_{i-1}$ and $q_i > k_{i-1}$ such that

$$\left(n_i! \sum_{k=k_{i-1}+1}^{k_i} |a_{n_ik}| (M+i)^{-1/p_k}\right)^{1/n_i} > i + C_{k_{i-1}}.$$

Having fixed n_i, by (4) choose $k_i > q_i$ such that

$$\left(n_i! \sum_{k=k_i+1}^{\infty} |a_{n_ik}| (M+i)^{-1/p_k}\right)^{1/n_i} < \epsilon.$$

As above using (8), (10) and (9) we can show that $(n_i! \mid y_{n_i} \mid)^{1/n_i} > i - \epsilon$ Since ϵ is arbitrary $(n_i! \mid y_{n_i} \mid)^{1/n_i} \to \infty$ as $i \to \infty$. Hence $(y_n) \notin \chi$ so that condition (1) is necessary.

Corollary 1 (Sridhar 1979) — $p_k = 1$ gives $A \in (c_0, \chi)$ if and only if

$$(n! \sum_{k\geqslant 1} |a_{nk}|)^{1/n} \to 0 \text{ as } n \to \infty.$$

Corollary 2 — $p_k = 1/k$ gives $A \in (\Gamma, \chi)$ if and only if

$$(n! \sum_{k\geqslant 1} |a_{nk}| M^{-k})^{1/n} \to 0 \text{ as } n \to \infty \text{ for some integer } M > 1.$$

In a similar manner by taking for all n, $(x_k) \in l^{\infty}(p)$ as $x_k = \operatorname{sgn}(a_{nk}) M^{1/p_k}$ for $1 \leqslant k \leqslant k_i$; $i = 1, 2, 3, \ldots$ and using Lemma *B*, we can prove the following:

Theorem 2 — $A \in (l^{\infty}(p), \chi)$ if and only if

$$(n! \sum_{k\geqslant 1} |a_{nk}| M^{1/p_k})^{1/n} \to 0 \text{ as } n \to \infty \text{ for every integer } M > 1.$$

Corollary 3 — $p_k = 1$ gives $A \in (l^{\infty}, \chi)$ if and only if

$$(n! \sum_{k \geqslant 1} | a_{nk} |)^{1/n} \to 0 \text{ as } n \to \infty.$$

Corollary 4 — $p_k = 1/k$ gives $A \in (\Gamma^x, \chi)$ if and only if

$$(n! \sum_{k > 1} | a_{nk} | M^k)^{1/n} \to 0 \text{ as } n \to \infty \text{ for every integer } M > 1.$$

Similarly using Lemma D and taking for all n, $(x_k) \in l^p$ as

$x_k = | a_{nk} |^{q-1} \text{ sgn } (a_{nk})$ for $1 \leqslant k \leqslant k_i$: $(i = 1, 2, 3, \ldots)$ we can prove:

Theorem 3 — $A \in (l^p, \chi)$ if and only if

$$(n! \sum_{k \geqslant 1} | a_{nk} |^q)^{1/n} \to 0 \text{ uniformly in } k \text{ as } n \to \infty$$

where $p > 1$ and $p^{-1} + q^{-1} = 1$.

Theorem 4 — $A \in (l, \chi)$ if and only if

$$(n! \; | a_{nk} |)^{1/n} \to 0 \text{ uniformly in } k \text{ as } n \to \infty. \qquad \ldots(13)$$

PROOF : *Sufficiency* — Since $(x_k) \in l$, $\sum_{k \geqslant 1} | x_k |$ is convergent converging to L (say). Let (13) holds, then given $\epsilon > 0$, there exists a $n_0 \geqslant 1$ such that

$$(n! \; | a_{nk} |)^{1/n} < \epsilon/L \text{ for } n \geqslant n_0. \qquad \ldots(14)$$

Then

$$(n! \; | y_n |)^{1/n} \leqslant (n! \sum_{k \geqslant 1} | a_{nk} | \; | x_k |)^{1/n} < \frac{\epsilon}{L} . L^{1/n} \text{ for } n \geqslant n_0$$

$$< \epsilon. \qquad \text{[using } \ldots(14)]$$

Hence $(y_n) \in \chi$.

Necessity — If (13) does not hold, there exists subsequences (n_p) and (k_p) such that

$$(n_p ! \; | a_{n_p} k_p |)^{1/n_p} > \epsilon \text{ as } p \to \infty. \qquad \ldots(15)$$

Since the matrix (a_{nk}) is applicable to each member of l, $(a_{nk}) \in l^x = l^\infty$ (using Lemma *C*). Hence $| a_{nk} | \leqslant A'_n$ for all k and for each n so that

$$n! \; | a_{nk} | \leqslant A_n \text{ where } A_n = n! \; A'_n \text{ for all } k \text{ and for each } n. \qquad \ldots(16)$$

Since $x = e^k \in l$, $(y_n) = (a_{nk}) \in \chi$ so that $(n! \; | a_{nk} |)^{1/n} \to 0$ as $n \to \infty$ for each k. Hence

$$(n! \; | a_{nk} |)^{1/n} \leqslant B_k \text{ for all } n \text{ and for each } k. \qquad \ldots(17)$$

In (16) and (17) the sequences (A_n) and (B_k) are unbounded as seen below.

If (A_n) is not unbounded, there exists a M such that $(n! \mid a_{nk} \mid)^{1/n} \leqslant A_n \leqslant M$ which contradicts our asssumption (15). Hence the sequence (A_n) and similarly the sequence (B_k) are unbounded.

Choose $n = n_1$ and $k = k_1$ so that

$$(n_1! \mid a_{n_1k_1} \mid)^{1/n_1} > 2; \; A_{n_1} > 1 \text{ and } (n! \mid a_{nk_1} \mid)^{1/n} \leqslant B_{k_1} \text{ for all } n. \quad \ldots(18)$$

Then choose $n_2 > n_1$ and $k_2 > k_1$ such that

$$\left(\frac{n_2! \mid a_{n_2k_2} \mid}{2A_{n_1}}\right)^{1/n_2} > 3+B_{k_1}; \; A_{n_1} \leqslant A_{n_2}; \; (n! \mid a_{nk_2} \mid)^{1/n} \leqslant B_{k_2} \text{ for all } n. \quad \ldots(19)$$

Proceeding in this manner, let us suppose that n_{p-1} and k_{p-1} have been already chosen. Then we can find a $n_p > n_{p-1}$ and $k_p > k_{p-1}$ such that

$$\left.\begin{aligned} &\left(\frac{n_p! \mid a_{n_pk_p} \mid}{2^{p-1} A_{n_{p-1}}}\right)^{1/n_p} > (p+1) + \sum_{i=2}^{p} \frac{B_{k_{i-1}}}{(2^{i-2} A_{n_{i-2}})^{1/n_i}} \\ &A_{n_{p-1}} \leqslant A_{n_p}; \text{ and } (n! \mid a_{n_pk_p} \mid)^{1/n} \leqslant B_{k_p} \text{ for all } n \end{aligned}\right\} \quad \ldots(20)$$

From the above $1 < A_{n_1} \leqslant A_{n_2} \leqslant \ldots\ldots \leqslant A_{n_p} \leqslant \ldots\ldots$ ...(21)

With the help of the above, we shall construct a sequence $(x_k) \in l$ such that (y_n) has a subsequence for which $(n! \mid y_n \mid)^{1/n}$ is unbounded so that $(y_n) \notin \chi$.

Taking

$$\left.\begin{aligned} x_{k_p} &= \frac{\operatorname{sgn}(a_{n_pk_p})}{2^{p-1} A_{n_{p-1}}} \text{ for } p = 1, 2, 3 \ldots \\ \text{and} \quad x_k &= 0 \text{ for } k \neq k_1, k_2, \ldots, k_p, \ldots \end{aligned}\right\} \quad \ldots(22)$$

so that $(x_k) \in l$.

$$\begin{aligned} (n_1! \mid y_{n_1}) \mid^{1/n_1} &= (n_1! \mid \sum_{j \geqslant 1} a_{n_1k_j} x_{k_j} \mid)^{1/n_1} \\ &\geqslant (n_1! \mid a_{n_1k_1} x_{k_1})^{1/n_1} - [n_1! \mid a_{n_1k_2} x_{k_2} \mid + n_1! \mid a_{n_1k_3} x_{k_3} + \ldots]^{1/n_1} \\ &\geqslant (n_1! \mid a_{n_1k_1} \mid)^{1/n_1} - \left(\frac{n_1! \mid a_{n_1k_2} \mid}{2A_{n_1}} + \frac{n_1! \mid a_{n_1k_3} \mid}{2^2 A_{n_2}} + \ldots\right)^{1/n_1} \\ &> 2 - \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right] \text{[using (18), (16) and (21)]} \\ &> 1 \end{aligned}$$

$$
\begin{aligned}
(n_2! \mid y_{n_2} \mid)^{1/n_2} &\geqslant (n_2! \mid a_{n_2k_2}x_{k_2} \mid)^{1/n_2} - (n_2! \mid a_{n_2k_1}x_{k_1} \mid)^{1/n_2} \\
&\quad - [n_2! \mid a_{n_2k_3}x_{k_3} \mid + n_2! \mid a_{n_2k_4}x_{k_4} \mid + \ldots]^{1/n_2} \\
&\geqslant \left(\frac{n_2! \mid a_{n_2k_2} \mid}{2A_{n_1}}\right)^{1/n_2} - (n_2! \mid a_{n_2k_1} \mid)^{1/n_2} \\
&\quad - \left[\frac{n_2! \mid a_{n_2k_3} \mid}{2^2 A_{n_2}} + \frac{n_2! \mid a_{n_2k_4} \mid}{2^3 A_{n_3}} + \ldots\right]^{1/n_2} \\
&> 3 - \frac{1}{2^2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right] \quad \text{[using (19), (18), (16) and (21)]} \\
&> 2.
\end{aligned}
$$

Proceeding in the same manner, we can find a n_p such that $(n_p! \mid y_{n_p} \mid)^{1/n_p} > p$ so that $(n! \mid y_n \mid)^{1/n} \rightarrow \infty$ through a subsequence of values of (n). Hence $(y_n) \notin \chi$ so that condition (13) is necessary.

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