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DEGREE OF APPROXIMATION BY NÖRLUND MEANS OF THE FOURIER-LAGUERRE EXPANSION

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In this paper, the authors estimate the degree of approximation of the Laguerre expansion at the point $x = 0$ by Nörlund means.

1. INTRODUCTION

Let Σu_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants and let us write

$$P(n) = P_n, \text{ where } P_n = \sum_{k=0}^{n} p_k.$$

The sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} \frac{p_{n-k} S_k}{P_n} = \sum_{k=0}^{n} \frac{p_k S_{n-k}}{P_n},\ P_n \neq 0 \qquad \ldots(1.1)$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$.

Two important particular cases of the Nörlund means are

(i) harmonic mean, when $p_k = \dfrac{1}{k+1}$

(ii) Cesàro mean, when $p_k = \dbinom{k+\delta-1}{\delta-1}$, $\delta > 0$.

The Laguerre expansion of a function $f(x) \in L\,[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad \ldots(1.2)$$

where

$$a_n = \left\{\Gamma(\alpha+1)\binom{n+\alpha}{n}\right\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y)\, L_n^{(\alpha)}(y)\, dy \qquad \ldots(1.3)$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, w^n = (1 - w)^{-\alpha-1} \exp\left(-\frac{xw}{1-w}\right) \qquad \ldots(1.4)$$

and existence of the integral (1.3) is presumed.

We write

$$\phi(y) = \{\Gamma(\alpha + 1)\}^{-1} e^{-y}\, y^{\alpha} \{f(y) - f(0)\}. \qquad \ldots(1.5)$$

Gupta (1971) estimated the order of the function by Cesàro means of the series (1.2) at the point $x = 0$ after replacing the continuity condition in Szegö's theorem (1959) by a much lighter condition. He established the following theorem:

Theorem A — If

$$F(t) = \int_0^t \frac{|f(y)|}{y}\, dy = o\left\{\log \frac{1}{t}\right\}^{1+p},\ t \to 0,\ -1 < p < \infty \qquad \ldots(1.6)$$

and

$$\int_1^{\infty} e^{-y/2}\, y^{(3\alpha-3k-1)/3}\, |f(y)|\, dy < \infty \qquad \ldots(1.7)$$

then

$$\sigma_n^k(0) = o\,(\log n)^{p+1}$$

provided $k > \alpha + \frac{1}{2}$, $\alpha > -1$, $\sigma_n^k(0)$ being the nth Cesàro mean of order k.

Recently, we (Beohar and Jadiya 1980) have estimated the order of function by Cesàro mean $\sigma_n^k(0)$, for $k > \alpha > -1$. The theorem is as follow:

Theorem B — For $k > \alpha > -1$

$$\sigma_n^k(f, 0) = O\,(n^{-1/4}) + O\{\psi(1/n)\} \qquad \ldots(1.8)$$

provided

$$\int_0^t |df(y)| \leqq A\psi(1/t),\ 0 \leqq t \leqq w < \infty, \qquad \ldots(1.9)$$

$$\int_w^{\infty} e^{-y/2}\, y^{(6\alpha-6k-1)/12}\, |df(y)| < \infty \qquad \ldots(1.10)$$

and

$$\int_{w}^{\infty} e^{-y/2}\, y^{(6\alpha-6k-13)/12}\, |\, f(y)\, |\, dy < \infty \qquad \ldots(1.11)$$

where $\psi(t)$ is a positive increasing function such that

$$\int_{c/n}^{\delta} \frac{\psi(t)}{t^2}\, dt = O\,\{n\psi(1/n)\},\ n \to \infty. \qquad \ldots(1.12)$$

Denoting the harmonic means by $\{t_n\}$, Singh (1977) estimated the order of function by harmonic means of the series (1.2) at point $x = 0$ by weaker conditions than those of theorem A. He proved the following:

Theorem C — For $-\frac{5}{6} < \alpha < -\frac{1}{2}$

$$t_n(0) - f(0) = o\,\{\log n\}^{p+1},$$

provided that

$$\int_{t}^{\delta} \frac{|\, \phi(y)\, |}{y^{\alpha+1}}\, dy = o\,\{\log 1/t\}^{1+p},\ t \to 0,\ -1 < p < \infty \qquad \ldots(1.13)$$

δ is a fixed positive constant,

$$\int_{\delta}^{n} e^{y/2}\, y^{-(2\alpha+3)/4}\, |\, \phi(y)\, |\, dy = o\,\{n^{-(2\alpha+1)/4}\,(\log n)^{1+p}\} \qquad \ldots(1.14)$$

and

$$\int_{n}^{\infty} e^{y/2}\, y^{-1/3}\, |\, \phi(y)\, |\, dy = o\,\{(\log n)^{\,p+1}\},\ n \to \infty. \qquad \ldots(1.15)$$

Recently, Jayaswal (1978) generalized the result of Singh (1977) and proved the following :

Theorem D — If for fixed δ and $-\frac{5}{6} < \alpha < -\frac{1}{2}$

$$\phi(x) \equiv \int_{t}^{\delta} \frac{|\, \phi(x)\, |}{x^{\alpha+1}}\, dx = o\left\{\log \frac{1}{t}\right\}^{p},\ t \to 0 \qquad \ldots(1.16)$$

$$\int_{\delta}^{n} e^{x/2}\, x^{-(2\alpha+3)/4}\, |\, \phi(x)\, |\, dx = o\,(\log n)^{p} \qquad \ldots(1.17)$$

and

$$\int_{n}^{\infty} e^{x/2}\, x^{-1/3}\, |\, \phi(x)\, |\, dx = o\,(\log n)^{p} \qquad \ldots(1.18)$$

then

$$t_n(0) - f(0) = o\,(\log n)^p \qquad \ldots(1.19)$$

where $\{p_n\}$ is a positive non increasing sequence such that

$$\sum_{\nu=0}^{n} \frac{p_{n-\nu}}{\nu + 1} = O\left(\frac{P_n}{n}\right) \qquad \ldots(1.20)$$

and t_n is the Nörlund mean.

2. Main Result

The object of the present paper is threefold:

(i) We prove our theorem for the Nörlund mean which is more general than harmonic mean.

(ii) We employ a condition which is weaker than the condition (1.13) of Theorem C, (1.17) of Theorem D and without application of (1.20).

(iii) In our theorem the range of α is increased to $-1 < \alpha < -\frac{1}{2}$, which is more useful for application. In fact we prove the following:

Theorem — If $\{p_n)$ is a positive non-increasing sequence of real numbers such that

$$\phi\,(t) \equiv \int_0^t |\,\phi(y)\,|\,dy = o\,\{t^{\alpha+1}\,P(1/t),\ t \to 0 \qquad \ldots(2.1)$$

$$\int_w^n e^{y/2}\,y^{-(2\alpha+3)/4}\,|\,\phi(y)\,|\,dy = o\,\{n^{-(2\alpha-1)/4}P_n\} \qquad \ldots(2.2)$$

and

$$\int_n^\infty e^{y/2}\,y^{-1/3}\,|\,\phi(y)\,|\,dy = o\,(P_n),\ n \to \infty, \qquad \ldots(2.3)$$

then for

$$-1 < \alpha < -\tfrac{1}{2}$$

$$t_n(0) - f(0) = o\,(P_n)$$

t_n is the Nörlund mean of Laguerre expansion.

3. Proof of the Theorem

By the relation $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$ we have

$$S_n(0) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y)\, dy$$

$$= \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y)\, L_n^{(\alpha+1)}(y)\, dy,$$

therefore $t_n(0)$ is given by

$$(P_n)^{-1} \sum_{k=0}^{n} p_k \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y)\, L_{n-k}^{(\alpha+1)}(y)\, dy.$$

Using orthogonal property of Laguerre polynomials and (1.5) we get

$$t_n(0) - f(0) = (P_n)^{-1} \sum_{k=0}^{n} p_k \int_0^\infty \phi(y)\, L_{n-k}^{(\alpha+1)}(y)\, dy$$

$$= \int_0^{c/n} + \int_{c/n}^{w} + \int_{w}^{n} + \int_{n}^{\infty}$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.} \qquad \ldots(3.1)$$

Using orthogonal property and order estimates as given in Sezgö (1959), we get

$$I_1 = (P_n)^{-1} \sum_{k=0}^{n} p_k \,.\, O\,(n-k)^{\alpha+1} \int_0^{c/n} |\,\phi(y)\,|\, dy$$

$$= (P_n)^{-1} \,.\, P_n \,.\, O\,(n^{\alpha+1}) \,.\, o\,(n^{-\alpha-1} P_n)$$

$$= o\,(P_n), \text{ as } n \to \infty. \qquad \ldots(3.2)$$

Next,

$$I_2 = (P_n)^{-1} \sum_{k=0}^{n} p_k \,.\, O\,(n-k)^{(2\alpha+1)/4} \int_{c/n}^{w} y^{-(2\alpha+3)/4}\, |\,\phi(y)\,|\, dy$$

Now

$$\sum_{k=0}^{n} p_k (n-k)^{(2\alpha+1)/4} = \{ \sum_{k=0}^{[n/2]} + \sum_{[n/2]+1}^{n} \} p_k\,(n-k)^{(2\alpha+1)/4}$$

$$= \{n - [n/2]\}^{(2\alpha+1)/4} P_{[n/2]} + \{p_{[n/2]}\, n^{(2\alpha+5)/4}$$

$$= O\,\{P_n \,.\, n^{(2\alpha+1)/4}\}.$$

Therefore

$$I_2 = O\,(n^{(2\alpha+1)/4})\, [\{y^{-(2\alpha+3)/4}\, \Phi(y)\}_{c/n}^{w} + \int_{c/n}^{w} y^{-(2\alpha+7)/4}\, \Phi(y)\, dy\,]$$

(*equation continued on p.* 1119)

$$= O\,(n^{(2\alpha+1)/4})\,[O\,(1) + o\,(n^{-(2\alpha+1)/4}\,P_n) + \int_{c/n}^{w} y^{-(2\alpha+3)/4}\,P(1/y)\,dy]$$

$$= O\,(1) + o\,(P_n) + o\,(P_n)\,.\,n^{(2\alpha+1)/4} \int_{c/n}^{w} y^{(2\alpha-3)/4}\,dy$$

$$= O\,(1) + o\,(P_n)$$

$$= o\,(P_n). \qquad \ldots(3.3)$$

Next, I_3 can be written as

$$O\,(P_n)^{-1} \sum_{k=0}^{n} p_{n-k} \int_{w}^{n} e^{y/2}\,y^{-(2\alpha+3)/4}\,|\,\phi(y)\,|\,e^{y/2}\,y^{(2\alpha+3)/4}\,|\,L_n^{(\alpha+1)}(y)\,|\,dy$$

$$= O\,(P_n)^{-1} \sum_{k=0}^{n} p_{n-k}\,.\,O\,(n^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2}\,y^{-(2\alpha+3)/4}\,|\,\phi(y)\,|\,dy$$

$$= O\,(n^{(2\alpha+1)/4})\,.\,o\,(n^{-(2\alpha+1)/4}\,P_n)$$

$$= o\,(P_n). \qquad \ldots(3.4)$$

Finally, considering I_4 we get

$$I_4 = O\,(P_n)^{-1} \sum_{k=0}^{n} p_{n-k} \int_{n}^{\infty} e^{y/2}\,y^{-(3\alpha+5)/6}\,|\,\phi(y)\,|$$

$$\times\; e^{-y/2}\,y^{(3\alpha+5)/6}\,|\,L_n^{(\alpha+1)}\,(y)\,|\,dy$$

$$= O\,(P_n)^{-1} \sum_{k=0}^{n} p_{n-k}\,.\,O\,(k^{(\alpha+1)/2}) \int_{n}^{\infty} \frac{e^{y/2}\,y^{-1/3}\,|\,\phi(y)\,|}{y^{(\alpha+1)/2}}\,dy$$

$$= O\,(P_n)^{-1}\,.\,O\,(P_n)\,.\,O\,(n^{(\alpha+1)/2}\,.\,n^{-(\alpha+1)/2})\,.\,o\,(P_n)$$

Therefore,

$$I_4 = o\,(P_n). \qquad \ldots(3.5)$$

Thus, by virtue of (3.1), (3.2), ..., (3.5), our theorem is proved.

References

Beohar, B. K., and Jadiya, B. L. (1980). Degree of approximation by Cesàro mean of Fourier-Laguerre series. *Indian J. pure appl. Math.*, **11**, 1162–65.

Gupta, D. P. (1971). Degree of approximation by Cesàro mean of Fourier-Laguerre expansion. *Acta Sci. Math.*, **32**, 255–59.

Jayaswal, A. K. (1978). Approximation of a function by Nörlund mean of Fourier-Laguerre expansion. *Vijnan Shodh Bharti*, **1**, 33–41.

Singh, T. (1977). Degree of approximation by harmonic means of Fourier-Laguerre expansion. *Publ. Math. Debrecen*, **24**, 53–57.

Szegö, G. (1959). Orthogonal Polynomials. Colloq. Publ. Amer. Math. Soc., New York.