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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY MEANS OF A CONJUGATE SERIES

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In this paper the author has determined the degree of approximation of certain functions belonging to the Lip α class by Nörlund means.

§1. Let f be periodic with period 2π and integrable in the Lebesgue sense. Let its Fourier series be given by

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \ldots(1.1)$$

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx). \qquad \ldots(1.2)$$

Let $\{p_n\}$ be a sequence of positive constants such that

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \to \infty \text{ as } n \to \infty.$$

Let

$$t_n = \frac{p_n s_0 + p_{n-1} s_1 + p_{n-2} s_2 + \ldots + p_0 s_n}{P_n}. \qquad \ldots(1.3)$$

If $t_n \to S$ as $n \to \infty$, we say that the sequence $\{s_n\}$ is summable by Nörlund mean or summable (N, P_n) to S (Hardy 1949).

A function $f \in$ Lip α ($\alpha > 0$) if

$$f(x+h) - f(x) = O(|h|^{\alpha}) \text{ for } 0 < \alpha \leqslant 1. \qquad \ldots(1.4)$$

§2. The well known theorem on the degree of approximation of a function $f(x)$ belonging to the class Lip α by (C, δ) means of its Fourier series is given by Alexits (1961, p. 301). Sahney and Goel (1973) have extended the above result to (N, P_n) means of its Fourier series. Later on Chandra (1975) has also studied similar problem for Riesz (R, P_n) means of its Fourier series.

The object of this paper is to prove a result for the conjugate series (1.2) by using (N, P_n) means. Precisely, we prove the following theorem.

Theorem — If the sequence $\{p_n\}$ satisfies the following conditions*

$$n \mid p_n \mid < C \mid P_n \mid \qquad \ldots(2.1)$$

and

$$\sum_{k=1}^{n} k \mid p_k - p_{k-1} \mid < C \mid P_n \mid \qquad \ldots(2.2)$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function f with period 2π and belonging to the class of Lip α, $0 < \alpha < 1$ by (N, P_n) means of its conjugate series, is given by

$$\mid \tilde{f}(x) - \tilde{t}_n(x) \mid = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}\right)$$

where $\tilde{t}_n(x)$ are the (N, P_n) means of the series (1.2).

We shall require the following lemmas for the proof of the theorem.

Lemma 1 (Sahney and Goel 1973, Lemma 1) — If the sequence $\{p_n\}$ is positive and non-increasing then for $\alpha > 0$

$$\frac{1}{n^{\alpha}} \leqslant \frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}. \qquad \ldots(2.3)$$

Lemma 2 (McFadden 1942, Lemma 5.11) — If $\{p_n\}$ is nonnegative and non-increasing then for $0 \leqslant a \leqslant b \leqslant \infty$; $0 \leqslant t \leqslant \pi$ and any n,

we have

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqslant P\left(\frac{1}{t}\right). \qquad \ldots(2.4)$$

Proof of the Theorem — Since

$$\tilde{S}_k(x) - \tilde{f}(x) = -\frac{1}{\pi} \int_0^{\pi} \psi(t) \frac{\cos (k + \frac{1}{2}) t}{2 \sin \frac{1}{2} t} \, dt$$

where $\psi(t) = f(x + t) - f(x - t)$

*Compare Hille and Tamarkin (1932).

we have

$$\tilde{t}_n(x) - \tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \psi(t)\,\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\,\frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Therefore

$$|\tilde{f}(x) - \tilde{t}_n(x)| \leqslant \frac{1}{\pi}\int_0^{\pi}\frac{|\psi(t)|}{2\sin\frac{1}{2}t}\left|\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\cos(k+\tfrac{1}{2})t\right|dt$$

$$= \frac{1}{\pi}\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right)\frac{|\psi(t)|}{2\sin\frac{1}{2}t}\left|\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\cos(k+\tfrac{1}{2})t\right|dt$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_1 = O\left(\int_0^{\pi/n} t^{\alpha}\cdot\frac{1}{t}\,dt\right)$$

$$= O\left(\frac{1}{n^{\alpha}}\right)$$

$$= O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right) \text{ (by Lemma 1)}$$

Since $\dfrac{1}{\sin\frac{1}{2}t} = O\left(\dfrac{1}{t}\right)$

and $$\left|\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\cos(k+\tfrac{1}{2})t\right| \leqslant 1.$$

Also

$$I_2 = O\left(\frac{1}{P_n}\int_{\pi/n}^{\pi}\frac{t^{\alpha}}{t}\left|\sum_{k=0}^{n} p_{n-k}\cos(k+\tfrac{1}{2})t\right|dt\right)$$

$$= O\left(\frac{1}{P_n}\int_{\pi/n}^{\pi} t^{\alpha-1}P\left(\frac{1}{t}\right)dt\right) \text{ (by Lemma 2)}$$

(equation continued on p. 1123)

$$= O\left(\frac{1}{P_n}\int_{n/\pi}^{1/\pi}\left(\frac{1}{y}\right)^{\alpha+1} P(y)\,dy\right)$$

$$= O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right).$$

This completes the proof of the theorem.

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