

## EXPANSION FORMULAS FOR SRIVASTAVA POLYNOMIALS IN SERIES OF THE KONHAUSER BIORTHOGONAL POLYNOMIALS

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Expansion formulas for the polynomials of Srivastava (1972) in series of Konhauser biorthogonal polynomials are obtained. The corresponding expansion formulas are also derived for certain classes of special hypergeometric polynomials (including the classical orthogonal polynomials) considered by Srivastava (1972).

### 1. INTRODUCTION

Konhauser (1967) introduced the following pair of biorthogonal polynomials suggested by the Laguerre polynomials  $L_n^{(\alpha)}(x)$  :

$$Z_n^\alpha(x; k) = \frac{\Gamma(1 + \alpha + kn)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(1 + \alpha + kj)} \quad \dots(1.1)$$

and

$$Y_n^\alpha(x; k) = \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left[ \frac{e^{-xt}(1+t)^{\alpha+kn}}{(t^{k-1} + kt^{k-2} + \dots + k)^{n+1}} \right]_{t=0} \quad \dots(1.2)$$

where  $x$  is real,  $\alpha > -1$  and  $k$  is any positive integer. The polynomials given by (1.1) are of degree  $n$  in  $x^k$  and are called Konhauser biorthogonal polynomial set of the first kind. The second set of polynomials given by (1.2) that are of degree  $n$  in  $x$  are called Konhauser biorthogonal polynomial set of the second kind. These polynomials have been extensively studied by Carlitz (1968), Prabhakar (1970, 1971), Srivastava (1973), Karande and Thakare (1975), Patil and Thakare (1978), and Srivastava and Singh (1979).

Srivastava (1972) introduced and studied a polynomial set  $S_n^m(x)$  defined by

$$S_n^m(x) = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} x^j, \quad n = 0, 1, 2, \dots \quad \dots(1.3)$$

where  $m$  is an arbitrary positive integer and  $A_{n,j}$  are arbitrary constants, real or complex. If in (1.3) we select [Srivastava 1972, p. 4, eqn. (7)]

$$A_{n,j} = \frac{(\alpha + \beta + n + 1)_j (a_1)_j \dots (a_\sigma)_j}{(b_1)_j \dots (b_s)_j}, \quad n, j = 0, 1, 2, \dots \quad \dots(1.4)$$

where the  $a$ 's and  $b$ 's are independent of  $n$ , then (1.3) give a class of generalized hypergeometric polynomials in the form

$$S_n^m(x) = {}_{m+\sigma+1}F_s \left[ \begin{matrix} -\frac{n}{m}, \dots, \frac{-(n-m+1)}{m}, \alpha + \beta + n + 1, a_1, \dots, a_\sigma; \\ b_1, \dots, b_s; \end{matrix} \right] x^{m^m} \quad \dots(1.5)$$

which were considered by Srivastava [1972, p. 4, eqn. (8)].

The purpose of this note is to obtain expansions of the polynomials  $S_n^m(x)$  in series of Konhauser biorthogonal polynomial sets.

### 2. INTEGRALS

Our analysis requires the following integrals the first of which can be readily evaluated with the help of (1.1).

$$\int_0^\infty e^{-x} x^\beta Z_n^\alpha(x; k) dx = (1 + \alpha)_{kn} \Gamma(1 + \beta) {}_{1+k}F_k \left[ \begin{matrix} -n, \frac{1+\beta}{k}, \dots, \frac{\beta+k}{k}; \\ \frac{1+\alpha}{k}, \dots, \frac{\alpha+k}{k}; \end{matrix} \right] 1 \quad \dots(2.1)$$

$$\begin{aligned} & \int_0^\infty e^{-x} x^\beta Z_n^\alpha(x; k) S_N^m(x) dx \\ &= (1 + \alpha)_{kn} \Gamma(1 + \beta) \cdot \sum_{s=0}^n \frac{(-n)_s (1 + \beta)_{ks}}{(1 + \alpha)_{ks} s!} \\ & \quad \times \sum_{j=0}^{[n/m]} \frac{(-N)_{mj} (1 + \beta + ks)_j}{j!} A_{N,j} \quad \dots(2.2) \end{aligned}$$

where  $\text{Re}(\beta) > -1$ , and  $n, N = 0, 1, 2, \dots$

The integral (2.2) can be derived easily by using (2.1). We also need the following biorthogonality relation

$$\int_0^\infty e^{-x} x^\alpha Z_n^\alpha(x; k) Y_n^\alpha(x; k) dx = \frac{\Gamma(1 + \alpha + kn)}{n!} \delta_{mn} \quad \dots(2.3)$$

where  $\delta_{mn}$  is Kronecker's delta, and  $\text{Re}(\alpha) > -1$ .

### 3. EXPANSIONS

For the Srivastava polynomials  $S_n^m(x)$  defined by (1.3), let

$$x^\beta S_N^m(x) = \sum_{n=0}^\infty A_n Y_n^\alpha(x; k). \quad \dots(3.1)$$

Multiply both sides of (3.1) by  $e^{-x} x^\alpha Z_n^\alpha(x; k)$ . Integrate over interval  $(0, \infty)$ ; then using (2.2) and (2.3), we get

$$A_n = \frac{n! \Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha)} \sum_{s=0}^n \frac{(-n)_s (1 + \alpha + \beta)_{ks}}{(1 + \alpha)_{ks} s!} \\ \times \sum_{j=0}^{[N/m]} \frac{(-N)_{mj} (1 + \alpha + \beta + ks)_j}{j!} A_{N,j} \quad \dots(3.2)$$

provided that  $\text{Re}(\alpha + \beta) > -1, \text{Re}(\alpha) > -1$ .

Hence (3.1) becomes

$$x^\beta S_N^m(x) = \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha)} \sum_{n=0}^\infty n! \left[ \sum_{s=0}^n \frac{(-n)_s (1 + \alpha + \beta)_{ks}}{(1 + \alpha)_{ks} s!} \right. \\ \left. \times \sum_{j=0}^{[N/m]} \frac{(-N)_{mj} (1 + \alpha + \beta + ks)_j}{j!} A_{N,j} \right] Y_n^\alpha(x; k) \quad \dots(3.3)$$

provided that  $\text{Re}(\alpha + \beta) > -1, \text{Re}(\alpha) > -1$ .

For  $k = 1$  we shall get expansion formulas in terms of Laguerre polynomials.

By resorting to this method it is not hard to obtain the expansion of the function (3.1) in terms of the Konhauser biorthogonal polynomial set of the first kind.

4. PARTICULAR CASES

In view of (1.5) we obtain the following expansion formula for the generalized hypergeometric polynomials :

$$\begin{aligned}
 (A) \quad & x^{\beta} {}_{m+\sigma+1}F_{\delta} \left[ \begin{matrix} -\frac{N}{m}, \dots, -\frac{(N-m+1)}{m}, \lambda+\mu+N+1, a_1, \dots, a_{\sigma}; \\ b_1, \dots, b_{\delta}; \end{matrix} \right]_{xm^m} \\
 &= \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} n! \left[ \sum_{s=0}^n \frac{(-n)_s (1+\alpha+\beta)_{ks}}{(1+\alpha)_{ks} s!} \right. \\
 & \times {}_{m+\sigma+2}F_{\delta} \left[ \begin{matrix} -\frac{N}{m}, \dots, -\frac{(N-m+1)}{m}, \lambda+\mu+N+1, \\ \alpha+\beta+ks+1, a_1, \dots, a_{\sigma}; \\ b_1, \dots, b_{\delta}; \end{matrix} \right]_{m^m} \\
 & \qquad \qquad \qquad \times Y_n^{\alpha}(x; k). \qquad \dots(4.1)
 \end{aligned}$$

(B) In fact, using (4.1) with  $m = \delta = 1, \sigma = 0, b_1 = \lambda + 1$  and  $x = \frac{1}{2}(1 - z)$ , we get

$$\begin{aligned}
 & \left( \frac{1-z}{2} \right)^{\beta} \frac{N!}{(1+\lambda)_N} P_N^{(\lambda, \mu)}(z) \\
 &= \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left[ \sum_{s=0}^n \frac{n! (-n)_s (1+\alpha+\beta)_{ks}}{(1+\alpha)_{ks} s!} \right. \\
 & \times {}_3F_1 \left[ \begin{matrix} -N, 1+\alpha+\beta+ks, 1+\lambda+\mu+N; \\ 1+\lambda; \end{matrix} \right]_1 Y_n^{\alpha} \left( \frac{1-z}{2}; k \right) \\
 & \qquad \qquad \qquad \dots(4.2)
 \end{aligned}$$

where  $P_N^{(\lambda, \mu)}(z)$  are the Jacobi polynomials.

(C) If we let

$$A_{N,j} = \frac{\Gamma(\lambda + N + 1)}{\Gamma(\lambda + j + 1)}, \quad N, j = 0, 1, 2, \dots,$$

and  $m = 1$ , we get

$$\begin{aligned}
 x^\beta L_N^{(\lambda)}(x) &= \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha)} \sum_{n=0}^{\infty} \left[ \sum_{s=0}^n \frac{n! (-n)_s (1 + \alpha + \beta)_s}{(1 + \alpha)_s s!} \right. \\
 &\quad \left. \times \sum_{j=0}^N \frac{(-1)^j (1 + \alpha + \beta + ks)_j}{j!} \binom{N + \lambda}{N - j} \right] Y_n^\alpha(x; k).
 \end{aligned}$$

...(4.3)

(D) Also, when

$$A_{N,j} = 2^{N-2j}, \quad N, j = 0, 1, 2, \dots,$$

we have, with  $m = 2$ ,

$$\begin{aligned}
 z^{-N-2\beta} H_N(z) &= \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha)} \sum_{n=0}^{\infty} \left[ \sum_{s=0}^n \frac{n! (-n)_s (1 + \alpha + \beta)_{ks}}{(1 + \alpha)_{ks} s!} \right. \\
 &\quad \left. \sum_{j=0}^{[N/2]} \frac{(-N)_{2j} (1 + \alpha + \beta + ks)_j}{j!} 2^{N-2j} \right] Y_n^\alpha(z^{-2}; k).
 \end{aligned}$$

...(4.4)

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