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A NOTE ON ALMOST CONVERGENCE OF FUNCTIONS

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Lorentz gave the idea of almost convergence of sequences which was applied to Fourier series by Dayal (1968). The almost convergence of functions is recently defined by Ho (1975). We, in the present note, apply this concept to Fourier integrals.

§1. A function $f(t)$ defined for $t > 1$ is said to be almost convergent to s, if

$$\frac{1}{x}\int_a^{a+x} f(t)\,dt \to s, \text{ uniformly for } a \geqslant 1 \text{ and } x \to \infty \text{ (Ho 1975).}$$

We define that the integral $\int_0^\infty \psi(u)\,du$ is almost convergent to s, if

$$\frac{1}{x}\int_a^{a+x} \Psi(t)\,dt \to s, \text{ uniformly for } a \geqslant 1 \text{ and } x \to \infty,$$

where $\Psi(t) = \int_0^t \psi(u)\,du.$

Let $f(t) \in L(-\infty, \infty)$, then the Fourier integral at $t = x$ is given by

$$A(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt.$$

We set $\phi(t) = f(x+t) + f(x-t) - 2s$, where s is a function of x.

§2. Dayal (1968) has applied the concept of almost convergence to Fourier series. Here in this note we shall study the almost convergence of Fourier integrals. It may be remarked that whereas in sequences, almost convergent sequence is necessarily bounded, the almost convergent functions need not be bounded. We shall prove the following Theorem.

Theorem — Let $f(t) \in L(-\infty, \infty)$, then the Fourier integral

$$\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(x-t)\,dt$$

is almost convergent to s, if the following conditions are satisfied:

$$\int_0^\tau |\phi(t)|\,dt = o(\tau) \text{ as } \tau \to 0+, \qquad \ldots(2.1)$$

$$\int_{1/(a+x)}^{1/x} \frac{|\phi(t)|}{t}\,dt = o(1) \text{ uniformly for } a \geqslant 1. \qquad \ldots(2.2)$$

PROOF : Let $\psi(u) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos u(x-t)\,dt$, then

$$\psi(u) = \frac{1}{\pi}\int_1^\infty [f(x+t)+f(x-t)]\cos ut\,dt$$

$$+\frac{2s}{\pi}\int_0^1 \cos ut\,dt + \frac{1}{\pi}\int_0^1 \phi(t)\cos ut\,dt$$

$$= \Psi_1 + \Psi_2 + \Psi_3, \text{ (say)}.$$

Now since $f(t) \in L(-\infty, \infty)$,

$$\int_0^\lambda \Psi_1(u)\,du = \frac{1}{\pi}\int_1^\infty \frac{f(x+t)+f(x-t)}{t}\sin\lambda t\,dt \to 0 \text{ as } \lambda \to \infty$$

by Riemann-Lebesgue Theorem.

Therefore $\int_0^\infty \Psi_1(u)\,du = 0.$

Also $$\int_0^\infty \Psi_2(u)\,du = \frac{2s}{\pi}\int_0^\infty \frac{\sin u}{u}\,du = s.$$

Hence $\int_0^\infty \psi(u)\,du$ is almost convergent if $\int_0^\infty \Psi_3(u)\,du$ is almost convergent and to prove this it is enough to prove that

$$\frac{1}{x}\int_a^{a+x} du \int_0^u \frac{1}{\pi}\left(\int_0^1 \phi(t)\cos vt\, dt\right) dv$$

$$= \frac{1}{x}\int_a^{a+x} du\, \frac{1}{\pi}\int_0^1 \frac{\phi(t)\sin ut}{u}\, dt \text{ tends to 0 as } x \to \infty, \text{ uniformly for } a \geqslant 1.$$

Now by Fubini's Theorem we have

$$\frac{1}{x}\int_a^{a+x} du\, \frac{1}{\pi}\int_0^1 \frac{\phi(t)}{t}\sin ut\, dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{\phi(t)}{t}\, dt\, \frac{1}{x}\int_a^{a+x} \sin ut\, du$$

$$= \left[\int_0^{1/(a+x)} + \int_{1/(a+x)}^{1/x} + \int_{1/x}^{1}\right] \frac{1}{\pi x}\frac{\phi(t)}{t}\, dt \int_a^{a+x} \sin ut\, du$$

$$= I_1 + I_2 + I_3 \text{ (say).}$$

Now
$$I_1 = O\left[\frac{1}{x}\int_0^{1/(a+x)} |\,\phi(t)\,|\, \frac{1}{\pi}\left((a+x)^2 - a^2\right) dt\right]$$

$$= O(a+x)\int_0^{1/(a+x)} |\,\phi(t)\,|\, dt = o(1),$$

by hypothesis (2.1) of the Theorem.

Again
$$I_3 = \frac{1}{x\pi}\int_{1/x}^{1} \frac{\phi(t)}{t}\, dt \int_a^{a+x} \sin ut\, du$$

$$= \frac{1}{x\pi}\int_{1/x}^{1} \frac{\phi(t)}{t^2}\left[\cos at - \cos (a+x)\, t\right] dt$$

$$= O\left[\frac{1}{x}\int_{1/x}^{1} \frac{|\,\phi(t)\,|}{t^2}\, dt\right]$$

(*equations continued on p.* 1142)

$$= O\left\{\frac{1}{x}\left[\frac{g(t)}{t^2}\right]_{1/x}^{1} + \frac{2}{x}\int_{1/x}^{1}\frac{g(t)}{t^3}\,dt\right\}$$

where $$g(t) = \int_0^t |\phi(u)|\,du.$$

Thus $I_3 = o(1)$, uniformly with respect to $a \geqslant 1$.

Finally $$I_2 = \frac{1}{x\pi}\int_{1/(a+x)}^{1/x}\frac{\phi(t)}{t^2}[\cos at - \cos(a+x)t]\,dt$$

$$= \frac{2}{x\pi}\int_{1/(a+x)}^{1/x}\frac{\phi(t)}{t^2}\sin(a+\tfrac{1}{2}x)\,t\cdot\sin\tfrac{1}{2}xt\,dt.$$

$$= O\left[\int_{1/(a+x)}^{1/x}\frac{|\phi(t)|}{t}\,dt\right] = o(1),$$

by the hypothesis (2.2) of the theorem.

This proves the theorem.

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