Indian J. pure appl. Math., **12**(9) : 1133–1145, September 1981

SOME RADIUS OF CONVEXITY PROBLEMS II*

R. BHARATI†

The Ramanujan Institute, University of Madras, Madras 600005

(*Received* 12 *November* 1980; *after revision* 18 *April* 1981)

Let $L(A, B)$ denote the class of functions f analytic in the unit disc E, with $f(0) = 0 = f'(0) - 1$, satisfying the condition

$$\frac{z^2 f'(z)}{g(z)\ h(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

where $-1 \leqslant A < B \leqslant 1$ and w is a unit function. Subjecting g and h to various conditions, a few properties of the class $L(A, B)$ are investigated.

1. INTRODUCTION

In this paper we introduce a new class of functions whose definition is based on the representation in terms of unit functions. This class of functions for special choices of the parameters reduces to a well-known class of univalent functions. We derive certain estimates leading to the determination of radius of convexity, obtain distortion and rotation theorems and coefficient estimates.

Let $P(A, B)$ denote the class of functions p analytic in $E = \{z : | z | < 1\}$ and of the form

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},\ z \in E, \quad -1 \leqslant A < B \leqslant 1$$

where w is analytic in E with $w(0) = 0$ and $| w(z) | < 1$ in E. This class, introduced by Janowski (1973), is a subclass of functions with positive real part.

Definition 1.1 — Let $L(A, B)$ denote the class of functions f, analytic in E, normalised by the conditions $f(0) = 0$ and $f'(0) = 1$ such that

$$\frac{z^2 f'(z)}{g(z)\ h(z)} \in P(A, B) \qquad \text{...(1.1)}$$

where g and h are subjected to the conditions : (i) $g \in S^*_\beta$ and $h \in K$; (ii) $g, h \in K$; and (iii) g and h satisfy Re $(g(z)/z) > 0$ and Re $(h(z)/z) > 0$ for $z \in E$, where S^*_β and K are the classes of starlike functions of order β and convex functions respectively.

*This forms a part of the thesis approved by the University of Madras, Madras, for the degree of Doctor of Philosophy.

†Present address : Loyola College, Madras 600034

The motivation for the study of the class $L(A, B)$ stems from the following observation : when $A = -1$, $B = 1$, $h(z) = z$ and $g \in S^*_\beta$ condition (1.1) means that

$$\text{Re}\ \frac{zf'(z)}{g(z)} > 0, z \in E$$

and the class $L(A, B)$ then reduces to a subclass of close-to-convex functions of order 0 and type β introduced by Libera (1964).

Furthermore, for suitable choice of A and B in various ways, several well-known subclasses can be realized as special cases. To be specific, if we denote by p, the function defined by

$$p(z) = z^2 f'(z)/g(z)h(z)$$

then

(a) $L(-\alpha, \alpha) = \{f : |(p(z) - 1)/(p(z) + 1)| < \alpha, \quad z \in E, 0 < \alpha \leqslant 1\}$

(b) $L(2\alpha - 1, 1) = \{f : \text{Re}\, p(z) > \alpha, \quad z \in E, 0 \leqslant \alpha < 1\}$

(c) $L\left(-1, 1 - \frac{1}{\alpha}\right) = \{f : |p(z) - \alpha| < \alpha, \quad z \in E, \alpha > \frac{1}{2}\}$

(d) $L(-\alpha, 0) = \{f : |p(z) - 1| < \alpha, \quad z \in E, 0 < \alpha \leqslant 1\}$

(e) $L\left(\frac{\alpha^2 - \beta^2 - \alpha}{\beta}, \frac{\alpha - 1}{\beta}\right) = \{f : |p(z) - \alpha| < \beta, \quad z \in E, 0 < \beta \leqslant \alpha\}$

(f) $L(2\alpha\beta - 1, 2\beta - 1)$

$= \{f : |(p(z) - 1)/\{2\beta(p(z) - \alpha) - (p(z) - 1)\}| < 1,$

$z \in E, 0 \leqslant \alpha < 1, 0 < \beta \leqslant 1\}$

(g) $L(-\beta, \alpha\beta) = \{f : |(p(z) - 1)/(\alpha p(z) + 1)| < \beta,$

$z \in E, 0 \leqslant \alpha \leqslant 1, 0 < \beta \leqslant 1\}$

Some of the results for the classes $L(-\alpha, \alpha)$, $L(2\alpha - 1, 1)$ and $L\left(-1, 1 - \frac{1}{\alpha}\right)$ appeared earlier in a paper of the author (Bharati 1978). Similar classes for different choices of p, were studied by Padmanabhan (1965, 1968), Robertson (1936), Singh and Goel (1971), Tuan and Anh (1976), Mogra and Juneja (1977) and Lakshminarasimhan (1977).

2. Inequalities for the Class $L(A, B)$

Theorem 2.1 — Let $f \in L(A, B)$, $g \in S^*_\beta$ and $h \in K$. Let $r(A, B)$ denote the smallest positive root, unique in (0, 1], of the equation

$$1 - 2r - (2B + 2A + AB + 1)\, r^2 - 2ABr^3 + ABr^4 = 0. \qquad \text{...(2.1)}$$

Then, for $|z| = r$, we have

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \begin{cases} \dfrac{1 + (2A + 2\beta - 2)\, r - [2(1-\beta)(B+A) + B - A - AB]\, r^2 - 2(1-\beta)\, ABr^3}{(1 + r)(1 + Ar)(1 + Br)}, \\ \qquad\qquad \text{for } 0 \leqslant r \leqslant r(A, B) \\ \dfrac{1 - 2(1-\beta)r}{1 + r} - \dfrac{B + A}{B - A} + \dfrac{2}{(B - A)(1 - r^2)} \\ \qquad \times \{\sqrt{(1 + A)(1 + B)(1 - Ar^2)(1 - Br^2)} - (1 - ABr^2)\}, \\ \qquad\qquad \text{for } r(A, B) \leqslant r < 1. \end{cases} \qquad \text{...(2.2)}$$

These estimates are sharp.

PROOF : $f \in L(A, B)$ means that

$$\frac{z^2 f'(z)}{g(z)\, h(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

where w is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. Differentiation yields

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} - 1 - \frac{(B - A)\, zw'(z)}{(1 + Aw(z))(1 + Bw(z))}. \qquad \text{...(2.3)}$$

Since w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$, we have

$$|zw'(z) - w(z)| \leqslant \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}. \qquad \text{...(2.4)}$$

Using well-known distortion theorems and inequality (2.4) in (2.3) we obtain, for $|z| = r$,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \frac{1 - 2(1-\beta)r}{1 + r} - (B - A)\left\{\operatorname{Re}\frac{w(z)}{(1 + Aw(z))(1 + Bw(z))} + \frac{r^2 - |w(z)|^2}{(1 - r^2)\,|1 + Aw(z)|\,|1 + Bw(z)|}\right\}.$$

Following the technique devised by Singh and Goel (1971), we consider the transformation $z \to p(z)$ given by

$$p(z) = (1 + Aw(z))/(1 + Bw(z))$$

which maps the region $|w(z)| \leqslant r$ onto $|p(z) - a| \leqslant d$ where

$$a = (1 - ABr^2)/(1 - B^2r^2) \text{ and } d = (B - A)\,r/(1 - B^2r^2).$$

Then we see that

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \frac{1 - 2(1-\beta)r}{1+r} - \frac{B+A}{B-A} + \frac{1}{B-A}\left\{\operatorname{Re}\,(Bp(z)) + \frac{A}{p(z)} - \frac{r^2\,|\,Bp(z)-A\,|^2 - |\,1-p(z)\,|^2}{(1-r^2)\,|\,p(z)\,|}\right\}. \qquad \ldots(2.5)$$

Put $p(z) = a + u + iv$, $R = |\,p(z)\,|$ and denote by $S(u, v)$ the right-hand side of (2.5). Then

$$S(u, v) = \frac{1 - 2(1-\beta)r}{1+r} - \frac{B+A}{B-A} + \frac{1}{B-A}\left\{B(a+u) + \frac{A(a+u)}{R^2} - \frac{(1 - B^2r^2)}{1-r^2}\frac{d^2 - u^2 - v^2}{R^4}\right\}. \qquad \ldots(2.6)$$

Differentiating (2.6) with respect to v,

$$\frac{\partial}{\partial v} S(u, v) = \frac{1}{B-A}\,\frac{V}{R^4}\,T(u, v)$$

where

$$T(u, v) = \frac{1 - B^2r^2}{1 - r^2}\,[2R^3 + (d^2 - u^2 - v^2)R] - 2A(a + u)$$

$$> 0.$$

So the minimum of $S(u, v)$ on every chord $u =$ constant, is attained when $v = 0$. Therefore the minimum of $S(u, v)$ in the disc $|\,p(z) - a\,| \leqslant d$ is attained on the diameter $v = 0$. On putting $v = 0$ in $S(u, v)$ we get

$$S(u, 0) \equiv L(R) = \frac{1 - 2(1-\beta)\,r}{1+r} - \frac{B+A}{B-A} + \frac{1}{(B-A)\,(1-r^2)} \times \left\{(1+B)\,(1 - Br^2)\,R + (1+A)\,(1 - Ar^2)\,\frac{1}{R} - 2a(1 - B^2r^2)\right\}$$

where $a - d \leqslant R \leqslant a + d$.

$L'(R) = 0$ when $R^2 = R_0^2 = (1 + A)(1 - Ar^2)/(1 + B)(1 - Br^2)$ and the absolute minimum equals

$$L(R_0) = \frac{1-2(1-\beta)r}{1+r} - \frac{B+A}{B-A} + \frac{2}{(B-A)(1-r^2)}$$
$$\times\{\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)} - (1-ABr^2)\}. \quad \ldots(2.7)$$

It is easy to show that $R_0 < a + d$. But R_0 may not always be greater than $a - d$. In such a case the minimum is attained at

$$R_1 = a - d = (1 + Ar)/(1 + Br) \text{ and}$$

$$L(R_1) = \frac{1+(2A+2\beta-2)r-[2(1-\beta)(A+B)+B-A-AB]r^2-2(1-\beta)ABr^3}{(1+r)(1+Ar)(1+Br)}. \quad \ldots(2.8)$$

The two minima given by (2.7) and (2.8) become equal to each other for such values of A and B for which $R_0 = R_1$. That is

$$\frac{(1+A)(1-Ar^2)}{(1+B)(1-Br^2)} = \left\{\frac{1+Ar}{1+Br}\right\}^2$$

which reduces to the following

$$\phi(A, B, r) = -1 + 2r + (2B + 2A + AB + 1)r^2 + 2ABr^3 - ABr^4$$
$$= 0.$$

Now $\phi(A, B, r)$ is a strictly increasing function of r, $0 \leqslant r < 1$, for $-1 \leqslant A < B \leqslant 1$ and $\phi(A, B, 0) = -1$ and

$$\phi(A, B, 1) = 2(1 + A)(1 + B) > 0.$$

Thus $\phi(A, B, r) = 0$ has a unique root $r(A, B)$ in $(0, 1]$. This gives the required estimates.

The first inequality in (2.2) is attained for $g(z) = z(1 + z)^{2\beta-2}$, $h(z) = z(1 + z)^{-1}$ and $w(z) = z$ and the second is attained for the same choice of g and h with $w(z) = z(z - t)/(1 - zt)$ where $|t| < 1$ and is determined by

$$\frac{1-(1+A)rt+Ar^2}{1-(1+B)rt+Br^2} = R_0.$$

Theorem 2.2 — Let $f \in L(A, B)$, $g \in S^*_\beta$ and $h \in K$. Then f is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation

(i) $1 + (2A + 2\beta - 2)r - [2(1-\beta)(A + B) + B - A - AB]r^2$
$- 2(1-\beta)ABr^3 = 0$, if $0 \leqslant r_0 \leqslant r(A, B)$.

(ii) $-4(1 + A) + 4(1 + A)(3 - 2\beta) r + [(B - A)(3 - 2\beta)^2$
$- 8(1 - \beta)(1 + A) + 4B(1 + A)] r^2 - 2(3 - 2\beta)[2(1 - \beta)(B - A)$
$+ B + A + 2AB] r^3 + [4(1 - \beta)^2(B - A) + 4(1 - \beta)(B + A + 2AB)$
$+ B - A] r^4 = 0$, if $r(A, B) \leqslant r_0 < 1$.

This result is sharp.

PROOF : By Theorem 2.1, f is convex if $\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant 0$. That is, if

$$1 + (2A + 2\beta - 2) r - [2(1 - \beta)(B + A) + B - A - AB] r^2 - 2(1 - \beta) ABr^3 \geqslant 0 \quad ...(2.9)$$

and

$$\frac{1 - 2(1 - \beta) r}{1 + r} - \frac{B + A}{B - A} + \frac{2}{(B - A)(1 - r^2)} \times \{\sqrt{(1 + A)(1 + B)(1 - Ar^2)(1 - Br^2)} - (1 - ABr^2)\} \geqslant 0. \quad ...(2.10)$$

(2.9) is valid only when $0 \leqslant r \leqslant r_0 \leqslant r(A, B)$ and (2.10) is valid only when $r(A, B) \leqslant r \leqslant r_0 < 1$.

Similar methods yield the following theorems.

Theorem 2.3 — Let $f \in L(A, B)$; $g, h \in K$ and $r(A, B)$ be the unique root, in (0, 1], of eqn. (2.1). Then, for $|z| = r$, we have

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \begin{cases} \dfrac{1 + (2A - 1) r + (AB - 2B) r^2 - ABr^3}{(1 + r)(1 + Ar)(1 + Br)}, & 0 \leqslant r \leqslant r(A, B) \\ \dfrac{1 - r}{1 + r} - \dfrac{B + A}{B - A} + \dfrac{2}{(B - A)(1 - r^2)} \\ \quad \times\{\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)} - (1 - ABr^2)\}, & r(A, B) \leqslant r < 1. \end{cases}$$

The bounds are sharp.

Theorem 2.4 — Let $f \in L(A, B)$. Suppose that $\text{Re}\,(g(z)/z) > 0$ and $\text{Re}\,(h(z)/z) > 0$ in E. Then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geqslant \begin{cases} \dfrac{1+(2A-4)r+(AB-4A-4B-1)r^2-2A(1+2B)r^3-ABr^4}{(1-r^2)(1+Ar)(1+Br)}, \\ \hfill 0 \leqslant r \leqslant r(A, B) \\ \dfrac{1-4r-r^2}{1-r^2} - \dfrac{B+A}{B-A} + \dfrac{2}{(B-A)(1-r^2)} \\ \times\{\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)} - (1-ABr^2)\}, \\ \hfill r(A, B) \leqslant r < 1, \end{cases}$$

where $r(A, B)$ is as defined in Theorem 2.1.

These estimates are sharp.

Analogues of Theorem 2.2 can also be obtained under the assumptions of Theorems 2.3 and 2.4.

3. Coefficient Estimates

Lemma 3.1 — Let $H \in P(A, B)$. If $H(z) = 1 + \sum_{k=1}^{\infty} H_k z^k$ then for $n \geqslant 1$,

$$| H_n | \leqslant B - A.$$

The bound is sharp for the function

$$H_0(z) = (1 - Az^n)/(1 - Bz^n).$$

Proof : Since $H \in P(A, B)$ we have

$$H(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in E \qquad \text{...(3.1)}$$

where w is analytic in E, $w(0) = 0$ and $| w(z) | < 1$ in E. From (3.1) it follows that

$$w(z)(BH(z) - A) = 1 - H(z).$$

Adopting the method of Clunie (1959), for $w(z) = \sum_{k=1}^{\infty} w_k z^k$, we get

$$\{B - A + B \sum_{k=1}^{\infty} H_k z^k\} \sum_{k=1}^{\infty} w_k z^k = - \sum_{k=1}^{\infty} H_k z^k. \qquad \text{...(3.2)}$$

Equating coefficients on both sides of (3.2) we obtain

$$(B - A)w_1 = -H_1$$

$$(B - A)w_n + B(H_1 w_{n-1} + \dots + H_{n-1} w_1) = -H_n.$$

It follows directly from (3.2) that

$$\{B - A + B \sum_{k=1}^{n-1} H_k z^k\} \sum_{k=1}^{\infty} w_k z^k = - \sum_{k=1}^{n} H_k z^k + \sum_{k=n+1}^{\infty} c_k z^k$$

where c_k's are some complex numbers. Then, since $| w(z) | < 1$, we have

$$\left| - \sum_{k=1}^{n} H_k z^k + \sum_{k=n+1}^{\infty} c_k z^k \right| < \left| B - A + B \sum_{k=1}^{n-1} H_k z^k \right|.$$

This yields

$$\sum_{k=1}^{n} | H_k |^2 + \sum_{k=n+1}^{\infty} | c_k |^2 < (B - A)^2 + B^2 \sum_{k=1}^{n-1} | H_k |^2.$$

Hence

$$| H_n |^2 \leqslant (B - A)^2 - (1 - B^2) \sum_{k=1}^{n-1} | H_k |^2$$

or

$$| H_n | \leqslant B - A.$$

The function $H_0(z) = (1 - Az^n)/(1 - Bz^n)$ shows that the bound is sharp.

Theorem 3.1 — Let $f \in L(A, B)$, $g \in S_0^*$ and $h \in K$. If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then for $n \geqslant 1$,

$$n | a_n | \leqslant \frac{n(n+1)}{2} + (B - A) \sum_{k=1}^{n-1} \frac{k(k+1)}{2}.$$

PROOF : Since $f \in L(A, B)$ we have

$$\frac{z^2 f'(z)}{g(z)\, h(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} = 1 + \sum_{k=1}^{\infty} H_k z^k. \qquad \ldots(3.3)$$

Setting $g(z) = \sum_{k=1}^{\infty} g_k z^k$, $g_1 = 1$ and $h(z) = \sum_{k=1}^{\infty} h_k z^k$, $h_1 = 1$ we have

$$g(z)\, h(z) = \sum_{k=1}^{\infty} (g_k h_1 + \ldots + g_1 h_k) z^{k+1}$$

$$= \sum_{k=1}^{\infty} A_{k+1} z^{k+1}, \text{ say.}$$

Using the well-known coefficient estimates for starlike and convex functions, we obtain

$$| A_{k+1} | \leqslant \frac{k(k+1)}{2}, \; k = 1, 2, \dots .$$

We therefore get from (3.3)

$$\sum_{k=1}^{\infty} k a_k z^{k+1} = \left\{ 1 + \sum_{k=1}^{\infty} H_k z^k \right\} \left\{ \sum_{k=1}^{\infty} A_{k+1} z^{k+1} \right\}$$
$$= A_2 z^2 + (A_3 + A_2 H_1) z^3 + \dots$$
$$+ (A_{n+1} + A_n H_1 + \dots + A_2 H_{n-1}) z^{n+1} + \dots .$$

Comparing coefficients,

$$n a_n = A_{n+1} + A_n H_1 + \dots + A_2 H_{n-1}.$$

Hence on using Lemma 3.1,

$$n \, | a_n | \leqslant \frac{n(n+1)}{2} + (B - A) \sum_{k=1}^{n-1} \frac{k(k+1)}{2}.$$

Similarly we prove the following theorems.

Theorem 3.2 — Let $f \in L(A, B)$ and $g, h \in K$. Then, for $n \geqslant 1$,

$$| a_n | \leqslant 1 + (B - A)(n - 1)/2.$$

Theorem 3.3 — Let $f \in L(A,B)$. Suppose that g and h satisfy $\operatorname{Re}(g(z)/z) > 0$ and $\operatorname{Re}(h(z)/z) > 0$ in E. Then, for $n \geqslant 1$,

$$n \, | a_n | \leqslant 4(n - 1) + (B - A) + 2(B - A)(n - 2)(n - 1).$$

4. Distortion Theorems

Theorem 4.1 — Let $f \in L(A, B)$, $g \in S^*_\beta$ and $h \in K$. Then, for $| z | = r$, we have

$$\frac{1 + Ar}{(1 + Br)(1 + r)^{3-2\beta}} \leqslant | f'(z) | \leqslant \frac{1 - Ar}{(1 - Br)(1 - r)^{3-2\beta}}.$$

The bounds are sharp.

Proof : The functions $p(z) = (1 + Aw(z))/(1 + Bw(z))$ are subordinate to $p_0(z) = (1 + Az)/(1 + Bz)$ which map the region $| w(z) | \leqslant r$ onto the disc with the line joining $(1 + Ar)/(1 + Br)$ and $(1 - Ar)/(1 - Br)$ as diameter. Hence

$$\frac{1 + Ar}{1 + Br} \leqslant | p(z) | \leqslant \frac{1 - Ar}{1 - Br} .$$

Since $f \in L(A, B)$, taking $p(z) = z^2 f'(z)/g(z)\, h(z)$ and using well-known distortion theorems we obtain the desired estimates.

Similarly we obtain the following theorems.

Theorem 4.2 — Let $f \in L(A, B)$ and $g, h \in K$. Then, for $|z| = r$,

$$\frac{1 + Ar}{(1 + Br)(1 + r)^2} \leqslant |f'(z)| \leqslant \frac{1 - Ar}{(1 - Br)(1 - r)^2}.$$

The bounds are sharp.

Theorem 4.3 — Let $f \in L(A, B)$ and assume that $\operatorname{Re}(g(z)/z) > 0$ and $\operatorname{Re}(h(z)/z) > 0$ in E. Then, for $|z| = r$, we have the following sharp bounds:

$$\frac{(1 + Ar)(1 - r)^2}{(1 + Br)(1 + r)^2} \leqslant |f'(z)| \leqslant \frac{(1 - Ar)(1 + r)^2}{(1 - Br)(1 - r)^2}$$

5. Rotation Theorems

Theorem 5.1 — Let $f \in L(A, B)$, $g \in S_\beta^*$ and $h \in K$. Then, for $|z| = r$, we have

$$|\arg f'(z)| \leqslant (3 - 2\beta) \arcsin r + \arcsin \frac{(B - A)\, r}{1 - ABr^2}.$$

The result is sharp.

Proof: If $f \in L(A, B)$, then

$$\frac{z^2 f'(z)}{g(z)\, h(z)} = p(z) \qquad \text{...(5.1)}$$

where $p(z) = (1 + Aw(z))/(1 + Bw(z))$, w being a unit function. The values of $p(z)$ are contained in a circle of radius $(B - A)\, r/(1 - B^2 r^2)$ with centre at the point $(1 - ABr^2)/(1 - B^2r^2)$. Thus $|\arg p(z)|$ attains its maximum at points where a ray through the origin is tangent to the circle. That is

$$|\arg p(z)| \leqslant \arcsin \frac{(B - A)\, r}{1 - ABr^2}.$$

From (5.1), we see that

$$\arg f'(z) = \arg \frac{g(z)}{z} + \arg \frac{h(z)}{z} + \arg p(z).$$

The result is then immediate if we apply the well-known rotation theorems. The estimate is sharp for the choice $g(z) = z(1 - \epsilon_1 z)^{2\beta-2}$, $h(z) = z(1 + \epsilon_2 z)^{-1}$ and $p(z) = (1 + A\epsilon_3 z)/(1 + B\epsilon_3 z)$ with suitably chosen ϵ_i, $|\epsilon_i| = 1$, $i = 1, 2, 3$.

Similarly we obtain the following theorems.

Theorem 5.2 — Let $f \in L(A, B)$ and $g, h \in K$. Then, for $|z| = r$,

$$|\arg f'(z)| \leqslant 2 \arcsin r + \arcsin \frac{(B - A)r}{1 - ABr^2}.$$

The estimate is sharp.

Theorem 5.3 — Let $f \in L(A, B)$. Assume that $\mathrm{Re}(g(z)/z) > 0$ and $\mathrm{Re}(h(z)/z) > 0$ in E. Then

$$|\arg f'(z)| \leqslant \pi + \arcsin \frac{(B - A)r}{1 - ABr^2}, \ |z| = r.$$

The classes S_β^* and K are rotation and conjugation invariant (Silverman (1975)). Hence the class $L(A, B)$ is also rotation and conjugation invariant. That is, $f(z) \in L(A, B)$ if and only if $\epsilon\, \overline{f(\epsilon \bar{z})}$ is in $L(A, B)$.

MacGregor (1969) showed that the exact radius of univalence of convex combinations of functions belonging to a rotation and conjugation invariant class is given by the supremum of r for which $\mathrm{Re}\, f'(z) > 0$, $|z| < r$, where f varies over all functions in the class. Since $\mathrm{Re}\, f'(z) > 0$ if and only if $|\arg f'(z)| < \pi/2$, we have the following results.

Theorem 5.4 — Suppose that $f_i \in L(A, B)$, $i = 1, 2$, with the associated functions g_i and h_i satisfying the conditions (i) $g_i \in S_\beta^*$, $h_i \in K$ and (ii) g_i, $h_i \in K$ in E, $i = 1, 2, \ldots$. Then

$$\lambda f_1(z) + (1 - \lambda) f_2(z) \quad (0 \leqslant \lambda \leqslant 1)$$

is univalent in a disc $|z| < r_0$, where r_0 is the smallest positive root of the equation

(i)′ $(3 - 2\beta) \arcsin r + \arcsin [(B - A) r/(1 - B^2 r^2)] = \dfrac{\pi}{2}$

(ii)′ $2 \arcsin r + \arcsin [(B - A) r/(1 - B^2 r^2)] = \dfrac{\pi}{2}$

respectively. The bounds in (i)′ and (ii)′ are sharp.

We conclude this paper by proving certain theorems on the partial sums of functions belonging to the class $L(A, B)$.

6. Partial Sums

Lemma 6.1 (Ruscheweyh 1977) — The function H analytic in E defined by

$$H(z) = \sum_{k=1}^{n} z^k$$

is prestarlike of order α, $\alpha \leqslant 1$ for $|z| < \dfrac{1}{4 - 2\alpha}$.

Theorem 6.1 — Let $f \in L(A, B)$, $g \in S_\beta^*$ and $h \in K$. Then we can write

$$\frac{zf'_n(z)}{G_n(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

the representation being valid for $|z| < 1/(5 - 2\beta)$, and w is a unit function.

Note : By $G_n(z)$ we mean the nth partial sum of $g(z)\ h(z)/z$ and by $f_n(z)$ that of $f(z)$.

PROOF : If $g \in S_\beta^*$ and $h \in K$, the function G defined by

$$G(z) = g(z)\ h(z)/z$$

is starlike of order $\beta - \frac{1}{2}$.

By Lemma 6.1 the function H, where $H(z) = \sum_{k=1}^{n} z^k$ is prestarlike of order $\beta - \frac{1}{2}$ for $|z| < 1/(5 - 2\beta)$. Then writing $F(z) = z^2 f'(z)/g(z)\ h(z)$ we see that

$$\Delta F = \frac{H * GF}{H * G} = \frac{zf'_n(z)}{G_n(z)}.$$

lies in the closed convex hull of the range of F, by a Theorem of Rusheweyh (1977). This proves our claim.

Similarly we obtain the following

Theorem 6.2 — Let $f \in L(A, B)$, $g, h \in K$. Then

$$\frac{zf'_n(z)}{G_n(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

for $|z| < 1/4$, w is a unit function.

The proof is immediate if we note that $G(z) = g(z)\ h(z)/z$ is a starlike function.

ACKNOWLEDGEMENT

The author is extremely grateful to Prof. K. S. Padmanabhan for his kind and helpful guidance while preparing this paper. The author expresses her gratefulness to the referee also for his helpful comments on this paper.

REFERENCES

Bharati, R. (1978). Some radius of convexity problems. *Indian J. pure appl. Math.*, **9**, 1118–30.

Clunie, J. (1959). On meromorphic Schlicht functions. *J. Lond. Math. Soc.*, **34**, 215–16.

Janowski, W. (1973). Some extremal problems for certain families of analytic functions. *Bull. Acad. Polon. Sci.*, **21**, 17–25.

Lakshminarasimhan, T. V. (1977). On subclasses of functions starlike in the unit disc. *J. Indian Math. Soc.*, **41**, 233–43.

Libera, R. J. (1964). Some radius of convexity problems. *Duke Math. J.*, **31**, 143–58.

MacGregor, T. H. (1969). The univalence of a linear combinations of convex mappings. *J. Lond. Math. Soc.*, **44**, 210–12.

Mogra, M. L., and Juneja, O. P. (1977). Coefficient estimates for starlike functions. *Bull. Austr. Math. Soc.*, **16**, 415–25.

Padmanabhan, K. S. (1965). On the radius of univalence and starlikeness of certain analytic functions. *J. Indian Math. Soc.* **29**, 71–80.

——————(1968). On certain classes of starlike functions in the unit disc. *J. Indian Math. Soc.*, **32**, 89–103.

Robertson, M. S. (1936). On the theory of univalent functions. *Ann. Math.*, **37**, 374–408.

Ruschweyh, S. (1977). Linear operators between classes of prestarlike functions. *Comment. Math. Helvetici*, **52**, 497–509.

Silverman, H. (1975). Linear combinations of convex mappings. *Rocky Mountain J. Math.*, **5**, 629–32.

Singh, V., and Goel, R. M. (1971). On radii of convexity and starlikeness of some classes of functions. *J. Math. Soc. Japan*, **23**, 323–39.

Tuan, P. D., and Anh, V. V. (1976). Extremal problems for a class of regular functions and applications. *University of Tasmania, Technical Report No.* 85.