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INDUCED POTENTIAL PROBLEM IN THREE DIMENSIONS—III (MIXED BOUNDARY CONDITIONS)

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The incubator problem in biomechanics is studied with the help of a mathematical model. Steady-state temperature is sought inside a cube in the presence of a symmetrically placed sphere of different thermal conductivity than the surrounding medium. The sides of the cube are insulated while the upper and lower faces are kept at different temperatures.

A generalized Green's function is constructed by the classical method of images. The discontinuity caused by the presence of the sphere is viewed as due to a single layer distribution on its surface and considered as an induced potential problem.

The density of the single layer satisfies a two-dimensional Fredholm integral equation of the second kind which may be solved by the well known numerical technique of approximating the kernel and forming a system of algebraic equations.

It is observed that the densities fall, though not appreciably, with respect to the rotation angle, while the temperature is independent of it. However, the latter increases near the sides of the cube and approaches the mean value close to the central plane.

1. INTRODUCTION

A mathematical model is made to study the incubator problem in biomechanics. We seek a steady-state temperature distribution inside a cube in the presence of a sphere of different thermal conductivity than the surrounding medium. The sphere is symmetrically placed inside it. The upper and lower faces of the cube are held at different temperatures while the side faces are insulated.

A generalized Green's function that conforms to the above mixed boundary conditions (Howland 1955) is generated by the classical method of images. The problem is viewed as one of induced potential (Birkhoff 1954) and the discontinuity introduced by the presence of the sphere is considered equivalent to a simple distribution on its surface. A two-dimensional Fredholm integral equation of the second kind (Fredholm 1900, 1903) gives the density of this distribution.

The integral equation is solved by the well known method of reduction to simultaneous algebraic equations by employing polynomial approximations of the kernel (Kantorovich and Krylov 1958), expressions for temperature distribution inside, on and outside the sphere may than be easily obtained and numerical verification of the boundary conditions and Plemelj's formulae (Kellogg 1929) carried out.

This essentially extends the work of Howland (1964) and gives a three dimensional analogue of the problem done for the planar figure.

2. Formulation of the Problem

Let S be a cube of unit dimensions, its edges being parallel to the co-ordinate axes, C be the surface of a sphere of radius $a < \frac{1}{2}$. Let R and R' denote regions interior and exterior ($\subset S$) to C that are filled with homogeneous and isotropic media of thermal conductivities σ_1 and σ_2 respectively.

Let T_1 and T_2 be the temperatures of the upper and the lower face of the cube given by $z = \pm \frac{1}{2}$. We assume that $T_1 \neq T_2 (T_1 > T_2)$. The side faces $x = \pm \frac{1}{2}$ and $y = \pm \frac{1}{2}$ are insulated. Let P, $\{P(x, y, z) : |x| \leqslant 1, |y| \leqslant 1, |z| \leqslant 1\} \in S$. In spherical polar co-ordinates (r, θ, φ), we have

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

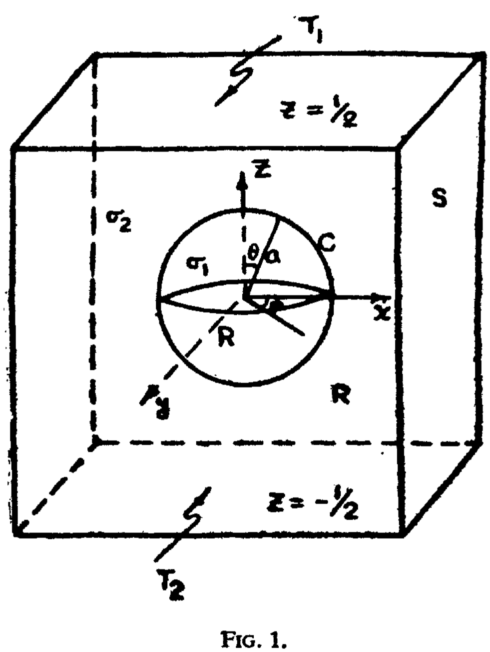

Fig. 1.

Let $\Phi(P)$ be the potential of the applied field due to the temperature T_1 and T_2. It is harmonic throughout the interior of S and satisfies the prescribed boundary conditions. Obviously,

$$\Phi(P) = \alpha z + \tfrac{1}{2}(T_1 + T_2) \qquad \ldots(2.1)$$

where $\alpha = T_1 - T_2$, represents the temperature distribution which would exist if the whole space were filled with an homogeneous and isotropic medium ($\sigma_1 = \sigma_2$).

Let $V(P)$ be the induced potential representing the effect of discontinuity across C. It has the following properties:

$$\left.\begin{array}{ll} \text{(i)} & \nabla^2 V = 0 \text{ in } R \text{ and } R' \\ \text{(ii)} & V(P) = 0 \text{ on } z = \pm \tfrac{1}{2} \\ \text{(iii)} & \dfrac{\partial V}{\partial x} = 0 \text{ on } x = \pm \tfrac{1}{2} \\ \text{(iv)} & \dfrac{\partial V}{\partial y} = 0 \text{ on } y = \pm \tfrac{1}{2}. \end{array}\right\} \qquad \ldots(2.2)$$

Then, the solution is given by the combined potential

$$U(P) = V(P) + \Phi(P) \qquad \ldots(2.3)$$

satisfying the following conditions:

$$\left.\begin{array}{ll} \text{(i)} & U(P) \text{ is continuous in the interior of } S \\ \text{(ii)} & \nabla^2 U = 0 \text{ in } R \text{ and } R' \\ \text{(iii)} & \sigma_1 \dfrac{\partial U}{\partial n_+} = \sigma_2 \dfrac{\partial U}{\partial n_-} \end{array}\right\} \qquad \ldots(2.4)$$

where $n_\pm$ indicate the inward and outward normals to C and the partial derivatives represent the limiting values of $\partial U/\partial n$ as C is approached along the normal, from within R and R' respectively. If we put $\lambda = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$, the condition (iii) above becomes

$$\frac{\partial V}{\partial n_-} - \frac{\partial V}{\partial n_+} = \lambda \left(\frac{\partial V}{\partial n_+} + \frac{\partial V}{\partial n_-} \right) + 2\lambda \frac{\partial \Phi}{\partial n}. \qquad \ldots(2.5)$$

From the above conditions we determine $V(P)$ and seek it as the potential of a single layer distribution on the surface of the sphere C.

Let $Q : \{Q(\xi, \eta, \zeta),\ |\xi| \leqslant \tfrac{1}{2}, |\eta| \leqslant \tfrac{1}{2},\ |\zeta| \leqslant \tfrac{1}{2} \in C\}$ have spherical polar co-ordinates

$$\left.\begin{array}{l} \xi = a \sin\theta' \cos\varphi' \\ \eta = a \sin\theta' \sin\varphi' \\ \zeta = a \cos\theta'. \end{array}\right\} \qquad \ldots(2.6)$$

If dS_Q be the surface element of C at Q, $H(P, Q)$, the generalized Green's function for the region and $\mu(Q)$, the density of the single layer, then the potential is given by

$$V(P) = \frac{1}{2\pi} \iint\limits_C H(P, Q)\, \mu(Q)\, dS_Q. \quad \ldots(2.7)$$

Evidently, $H(P, Q)$ satisfies the following conditions:

$$\left.\begin{array}{ll} \text{(i)} & \nabla^2 H = 0 \text{ both in } P \text{ and } Q, \text{ except for } P = Q \\ \text{(ii)} & H = 0 \text{ on } z = 0, \pm \tfrac{1}{2} \\ \text{(iii)} & \dfrac{\partial H}{\partial x} = 0 \text{ on } x = 0, \pm \tfrac{1}{2} \\ \text{(iv)} & \dfrac{\partial H}{\partial y} = 0 \text{ on } y = 0, \pm \tfrac{1}{2}. \end{array}\right\} \quad \ldots(2.8)$$

The potential $V(P)$ is continuous throughout interior of S and satisfies the following conditions:

$$\left.\begin{array}{ll} \text{(i)} & \nabla^2 V = 0 \text{ throughout } R \text{ and } R' \\ \text{(ii)} & V = 0 \text{ on } z = 0, \pm \tfrac{1}{2} \\ \text{(iii)} & \dfrac{\partial V}{\partial x} = 0 \text{ on } x = 0, \pm \tfrac{1}{2} \\ \text{(iv)} & \dfrac{\partial V}{\partial y} = 0 \text{ on } y = 0, \pm \tfrac{1}{2}. \end{array}\right\} \quad \ldots(2.9)$$

Since n, the normal to C coincides with the radial direction r, the discontinuity relations in the normal derivatives of (2.7) are given by (Kellogg 1929, p. 309)

$$\tfrac{1}{2}\left(\frac{\partial V}{\partial r_-} - \frac{\partial V}{\partial r_+}\right) = \mu(p) \quad \ldots(2.10)$$

and

$$\tfrac{1}{2}\left(\frac{\partial V}{\partial r_-} + \frac{\partial V}{\partial r_+}\right) = \iint\limits_C K(p, q)\, \mu(q)\, dS_q \quad \ldots(2.11)$$

where, $p(\theta, \varphi)$ and $q(\theta', \varphi')$ correspond to the points P and Q on C and

$$dS_q = \sin\theta'\, d\theta'\, d\varphi'.$$

$$K(p, q) = -\frac{a^2}{2\pi} \cdot \frac{\partial}{\partial r} H(P, Q)\Big|_{r=a}$$

$$= -\frac{a^2}{2\pi} \frac{\partial}{\partial r} H(P, q)\Big|_{r=a} \quad \ldots(2.12)$$

Substituting from (2.10) and (2.11) into (2.5), we see that μ satisfies the functional equation

$$\mu(p) = \lambda\alpha \cos\theta + \lambda \iint_C K(p, q) \cdot \mu(q)\, dS_q \qquad \text{...(2.13)}$$

which is Fredholm's integral equation of the second kind (Fredholm 1900, 1903). Properties of kernels of the type (2.12) have been discussed in detail (Nayar 1977, 1978). In order to find $K(p, q)$, we first construct the generalized Green's function $H(P, Q)$, by employing the classical method of images.

3. Analogue of Weierstrass' σ-Function

Let R represent a point with Cartesian co-ordinates (l_1, l_2, l_3), where l_i are each integers, positive, negative or zero such that $l_1 = l_2 = l_3 \neq 0$.

Let the space be divided into elementary cells by planes parallel to the co-ordinate planes and passing through the points

$$\pm \tfrac{1}{2}, \pm \tfrac{3}{2}, \pm \tfrac{5}{2}, \ldots$$

on each co-ordinate axis. Thus each point R is at the centre of an elementary cell of unit dimensions. We shall refer to these cells as 'layers' and 'cubes' as follows:

First cube is the cell that surrounds the origin, first layer will be those $(3^3 - 1)$ cells which completely surround the first cube, second cube will be the first cube together with the first layer. Second layer will consist of those $5^3 - 3^3$ cells that completely surround the second cube. Third cube will consist of second cube together with second layer i.e. 5^3 cells. Similarly, the nth cube will contain $(2n - 1)^3$ cells and the nth layer $[(2n - 1)^3 - (2n - 3)^3]$ cells.

Referring to the centre O as origin, let $OP = \rho$, $OR = L$ and $PR = r'$, then

$$\left.\begin{aligned} \rho &= (x^2 + y^2 + z^2)^{1/2} \\ L &= (l_1^2 + l_2^2 + l_3^2)^{1/2} \\ r' &= [(x - l_1)^2 + (y - l_2)^2 + (z - l_3)^2]^{1/2}. \end{aligned}\right\} \qquad \text{...(3.1)}$$

Let γ be the angle POR, then if u denotes $\cos\gamma$, we have

$$u = \cos\gamma = (xl_1 + yl_2 + zl_3)/\rho L. \qquad \text{...(3.2)}$$

We may expand $1/r'$ in terms of Legendre polynomials (Kellogg 1929, p. 124). Let $h = \rho/L$, then

$$\frac{1}{r'} = \frac{1}{L}\,[P_0(u) + P_1(u) \cdot h + P_2(u)\, h^2 + P_3(u)\, h^3 \ldots] \qquad \text{...(3.3)}$$

were $P_n(u)$ is Legendre polynomial of order n. The above expansion is valid (Kellogg 1929, p. 125) for $|\, h \,| < \sqrt{2} - 1$. Here,

$$P_0 = 1,\ P_1 = u,\ P_2 = \tfrac{3}{2}(u^2 - \tfrac{1}{3})\ \ldots, \quad u = \cos\gamma.$$

In general, $|P_n| \leqslant 1$.

For points P interior to C, since $|x| < \frac{1}{2}$, etc. we have $\rho^2 \leqslant \frac{3}{4}$ and $\rho < L$ whence $\rho \leqslant \sqrt{\frac{3}{2}}$ and $L > \sqrt{\frac{3}{2}}$. For R, in third and succeeding layers, this condition is satisfied, since $L \geqslant 3$.

We define an elementary function

$$f(P, R) = \frac{1}{r'} - \frac{1}{L}(P_0 + hP_1 + h^2P_2) \qquad \ldots(3.4)$$

which by virtue of (3.3) gives

$$f(P, R) = \frac{h^3}{L}(P_3 + hP_4 + \ldots). \qquad \ldots(3.5)$$

Using (3.2) and the value of h, (3.4) becomes in terms of the co-ordinates,

$$f(P, R) = \frac{1}{r'} - \frac{1}{L} - \frac{1}{L^3}(xl_1 + yl_2 + zl_3) - \frac{3}{2L^5}(xl_1 + yl_2 + zl_3)^2 + \tfrac{1}{2}\cdot\frac{1}{L^3}(x^2 + y^2 + z^2). \qquad \ldots(3.6)$$

The function f defined by (3.4) has following properties:

(i) $|f(P, R)|$ is bounded for R in the third layer. Since $|P_n| \leqslant 1$, we have from (3.5)

$$|f(P, R)| \leqslant \frac{h^3}{L}(1 + h + h^2 + \ldots)$$

$$= \frac{\rho^3}{L^3(L - \rho)} \leqslant \frac{9}{4L^4(2\sqrt{3} - 1)} \qquad \ldots(3.7)$$

where we have used the bounds on ρ and L above.

(ii) The series

$$\sigma_1(P) = \sum_R{}'' f(P, R) \qquad \ldots(3.8)$$

where $\sum_R''$ denotes summation for all R lying in the third and succeeding layers, is uniformly and absolutely convergent.

(iii) The series $\sum'' \frac{1}{L^5}$ is absolutely convergent.

(iv) The partial sum of the series (3.8) satisfy the conditions of Harnack's first theorem (Kellogg 1929, p. 248).

Thus $\sigma_1(P)$ is a harmonic function and we may compute its derivatives of any order by term wise differentiation. The series thus obtained is uniformly convergent.

We define $f(P, R)$ for R in the first and second layer by the relation (3.6) and for R at the origin by $f(P, 0) = 1/\rho$.

Let

$$\sigma(P) = \frac{1}{\rho} + \sum_R{}' f(P, R) \qquad \ldots(3.9)$$

where $\sum_R'$ means the summation on R over all points lying in first and subsequent layers, except the origin. It has the following properties:

(i) It is harmonic for all values of P except the origin.

(ii) Its derivatives of any order may be computed by term wise differentiation, the series thus obtained being uniformly and absolutely convergent. $\sigma(P)$ is the analogue of the Weierstrass σ-function in two dimensions (Howland 1964).

4. Method of Images : Generalized Green's Function

Employing the method of images and using mass points which would give rise to potentials with properties (2.8), we see that a combination of eight points is needed, four having positive unit mass and four having negative unit mass. In order that the potential on the xy plane be zero, negative masses are put at the mirror images, in the xy plane, of the points with positive masses.

Let the first cube surrounding the origin be divided into eight parts by the three co-ordinate planes. Let $Q(\xi, \eta, \zeta)$ be confined to one of the octants and $P(x, y, z)$ to the first cube, then $| x \pm \xi | \leqslant 1$, etc. Now we associate positive mass at the four points $(\pm \xi, \pm \eta, \zeta)$ and negative masses at the images, reflexions in $z = 0$ plane, at the four points $(\pm \xi, \pm \eta, -\zeta)$. Consider the function

$$H_1(P, Q) = \Sigma\sigma(x \pm \xi, y \pm \eta, z - \zeta) - \Sigma\sigma(x \pm \xi, y \pm \eta, z + \zeta). \qquad \ldots(4.1)$$

Here, summation extends over four points with arguments in each case as indicated above. Let S define the sum of eight terms with arguments and signs the same as the corresponding σ-function above.

Furthermore, co-ordinates of the principal mass point in the cell with centre (l_1, l_2, l_3) will be $(\xi + l_1, \eta + l_2, \zeta + l_3)$. Consequently (3.1) gives

$$r'^2 = [(x - \xi) - l_1]^2 + [(y - \eta) - l_2]^2 + [(z - \zeta) - l_3]^2. \qquad \ldots(4.2)$$

For a typical term, we have from (3.6) and (3.9), the modified σ-function as

$$\sigma(x-\xi, y-\eta, z-\zeta) = \frac{1}{\rho'} + \sum_R{}' \Big[\Big(\frac{1}{r'} - \frac{1}{L} - \frac{1}{L^3}$$

$$\times \{(x-\xi)\, l_1 + (y-\eta)\, l_2 + (z-\zeta)\, l_3\} - \frac{3}{2L^5}$$

$$\times [(x-\xi)\, l_1 + (y-\eta)\, l_2 + (z-\zeta)\, l_3]^2 + \frac{1}{2L^3}$$

$$\times \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}\Big)\Big] \qquad \ldots(4.3)$$

where

$$\rho'^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \qquad \ldots(4.4)$$

and r'^2 is given by (4.2).

Now we wish to sum, over eight terms of (4.1) with proper arguments and signs. The following results are needed in the sequel. Using the S-summation notation for such combinations, defined earlier, we have

$$S\,\frac{1}{L} = 0$$

$$\left.\begin{aligned} &S[(x-\xi)\, l_1 + (y-\eta)\, l_2 + (z-\zeta)\, l_3] = -8\zeta l_3 \\ &S(x-\xi)(y-\eta)\, l_1 l_2 = 0 \\ &S(x-\xi)(z-\zeta)\, l_1 l_3 = -8x\zeta l_1 l_3 \\ &S(y-\eta)(z-\zeta)\, l_2 l_3 = -8y\zeta l_2 l_3 \\ &S[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2] = -8z\zeta \\ &S[(x-\xi)^2\, l_1^2 + (y-\eta)^2\, l_2^2 + (z-\zeta)^2\, l_3^2] = -8z\zeta l_3^2 \\ &S[(x-\xi)\, l_1 + (y-\eta)\, l_2 + (z-\zeta)\, l_3]^2 \\ &\qquad\qquad = -16\zeta l_3(xl_1 + yl_2 + zl_3). \end{aligned}\right\} \qquad \ldots(4.5)$$

Consequently (4.3) and (4.5) yield

$$H_1(P, Q) = S\sigma(P) = S\,\frac{1}{\rho'} + \sum_R{}' \Big[S\,\frac{1}{r'} - \frac{8\zeta}{L^3}(z - l_3)$$

$$+ \frac{24}{L^5}\,\zeta l_3(xl_1 + yl_2 + zl_3)\Big]. \qquad \ldots(4.6)$$

Here Σ'_R implies summation over all positive and negative integral values of l_1, l_2 and l_3. Consequently, we have

$$\left.\begin{aligned} &\sum{}' \frac{\zeta l_3}{L^3} = 0 \\ &\sum{}' \frac{l_1 l_3}{L^5} = 0 \\ &\sum{}' \frac{l_2 l_3}{L^5} = 0. \end{aligned}\right\} \qquad \ldots(4.7)$$

Now let

$$H(P, Q) = S\frac{1}{\rho'} + \sum{}' \left[S\frac{1}{r'} - \frac{8z\zeta}{L^3} + \frac{24z\zeta l_3^2}{L^5} \right] - kz \qquad \ldots(4.8)$$

where k is a constant to be determined such that H has the requisite properties:

(i) H is an even function of x and y and an odd function of z.

(ii) H is periodic in x and y with period 1.

(iii) H_x and H_y are odd functions of x and y, respectively. Subscripts denote partial derivatives.

(iv) $H_x = 0$ at $x = 0, \pm \frac{1}{2}$

$H_y = 0$ at $y = 0, \pm \frac{1}{2}$

(v) H_{zz} is an odd function of z, and is periodic in x, y and z with period 1.

(vi) H_{xz} and H_{yz} are periodic in x, y and z with period 1.

(vii) $H(P, Q) = 0$, for all Q, whenever P lies on $z = 0, \pm \frac{1}{2}$.

The last condition gives $k = 0$ in (4.8) above. Hence the generalized Green's function becomes

$$H(P, Q) = S\frac{1}{\rho'} + \sum{}' \left(S\frac{1}{r'} - \frac{8z\zeta}{L^3} + \frac{24z\zeta l_3^2}{L^5} \right). \qquad \ldots(4.9)$$

5. Polynomial Approximation of the Kernel

We are now in a position to give a polynomial approximation of the kernel K given by (2.12). Since $x = r\sin\theta\cos\varphi$, we have $x_r = x/r$, etc, and from (4.4), partially differentiating with respect to r, we have

$$\rho'_r = \{(x - \xi)\,x + (y - \eta)\,y + (z - \zeta)\,z\}/r\rho\,. \qquad \ldots(5.1)$$

Similarly from (4.2)

$$r'_r = [\{(x - \xi) - l_1\}\,x + \{(y - \eta) - l_2\}\,y + \{(z - \zeta) - l_3\}\,z]/rr'. \qquad \ldots(5.2)$$

From (4.9), by inverting the order of differentiation and summation, we get

$$-a^2H_r = a^2\Big[S \frac{1}{\rho'^2}\, \rho'_r + \sum S \frac{1}{r'^2}\, r'_r + \sum \frac{8\zeta}{L^3}\, z_r$$
$$- \sum \frac{24\zeta l_3^2}{L^5}\, z_r\Big]. \qquad \ldots(5.3)$$

Substituting from (5.1) and (5.2) we obtain, for the kernel, the value of the following expression for $r = a$.

$$- a^2H_r = S \frac{a^2}{r} \frac{1}{\rho'^3} \{(x-\xi)\, x + (y-\eta)\, y + (z-\zeta)z\}$$
$$+ \sum{}' S \frac{a^2}{r} \cdot \frac{1}{r'^3} [\{(x-\xi) - l_1\}\, x + \{(y-\eta) - l_2\}\, y$$
$$+ \{(z-\zeta) - l_3\}\, z] + \sum{}' \frac{8a^2}{rL^3}\, z\zeta - \sum{}' \frac{24a^2}{rL^5}\, z\zeta l_3^2 \Big|_{r=a.} \quad (5.4)$$

To evaluate it, we first expand the various expressions in terms of Legendre polynomials, then sum over S and finally put $r = a$. We treat each term in (5.4) separately.

Let $h = a/r$ and $t = \cos\alpha = \dfrac{x\xi + y\eta + z\zeta}{ar}$, then for $|\,h\,| < 1$,

$$\Sigma\,(x-\xi)\, x = r^2(1 - ht) \qquad \ldots(5.5)$$

$$\rho'^{-3} = r^{-3}\,(1 - 2th + h^2)^{-3/2}$$

Recall the formula [Whittaker and Watson 1965, p. 329]

$$(1 - 2th + h^2)^{-m} = \Sigma\, h^n C_n^m(t),\ C_0^m = 1 \qquad \ldots(5.6)$$

where

$$C_0^{3/2} = 1,\ C_1^{3/2} = 3t,\ C_2^{3/2} = \tfrac{3}{2}(5t^2 - 1), \text{ etc. } \ldots \qquad \ldots(5.7)$$

Thus for $r > a$,

$$1/\rho'^3 = r^{-3}\{1 + 3ht + \tfrac{3}{2}h^2\,(5t^2 - 1) \ldots\}$$
$$= r^{-3}[1 + 3r^{-2}\,(x\xi + y\eta + z\zeta) + \tfrac{3}{2}\{5r^{-4}(x\xi + y\eta + z\zeta)^2 - a^2/r^2\} \ldots].$$
$$\ldots(5.8)$$

Similarly, we get the expansion for $r < a$, as

$$1/\rho'^3 = a^{-3}\,[1 + 3a^{-2}\,(x\xi + y\eta + z\zeta)$$
$$+ \tfrac{3}{2}\{5a^{-4}(x\xi + y\eta + z\zeta)^2 - r^2a^{-2}\}. \qquad \ldots(5.9)$$

Consequently (5.5) and (5.8) give the first term on the right of (5.4) for $r > a$ as

$$S\ \frac{a^2}{r^4}\left[(r^2-\tfrac{3}{2}a^2)+\left(2+\frac{3a^2}{2r^2}\right)(x\xi+y\eta+z\zeta)\right.$$
$$\left.+\frac{9}{2r^2}(x\xi+y\eta+z\zeta)^2-\frac{15}{2r^4}(x\xi+y\eta+z\zeta)^3\ldots\right]. \quad \ldots(5.10)$$

Summing over eight terms with arguments and signs as explained above and using the following simplifications:

$$\left.\begin{aligned} &S(1)=0\\ &S(x\xi+y\eta+z\zeta)=8z\zeta\\ &S(x\xi+y\eta+z\zeta)^2=0\\ &S(x\xi+y\eta+z\zeta)^3=8z\zeta\{3(x^2\xi^2+y^2\eta^2)+z^2\zeta^2\}, \end{aligned}\right\} \quad \ldots(5.11)$$

the first term of $K(P, Q)$ in (5.10) for $r > a$ becomes

$$=\frac{a^2}{r^4}\left[\left(2+\frac{3a^2}{2r^2}\right)8z\zeta-\frac{60}{r^4}\cdot z\zeta\{3(x^2\xi^2+y^2\eta^2)+z^2\zeta^2\}\ldots\right] \ldots(5.12)$$

For $r = a$, it is

$$=\frac{28z\xi}{a^2}-\frac{60}{a^6}z\zeta\{3(x^2\xi^2+y^2\eta^2)+z^2\zeta^2\}. \quad \ldots(5.13)$$

For interior points $r < a$, using (5.9) and (5.11), the first term becomes

$$=\frac{1}{ar}\left\{\left(\frac{9r^2}{2a^2}-1\right)8z\zeta-\frac{60}{a^4}z\zeta\{3(x^2\xi^2+y^2\eta^2)+z^2\zeta^2\}\right. \quad \ldots(5.14)$$

which reduces to (5.13) for $r = a$.

Now, for the second term using values of r', ρ', L, etc. we have,

$$1/r'^3 = L^{-3}(1-2t'h'+h'^2)^{-3/2} \quad \ldots(5.15)$$

where

$$h'=\rho'/L \text{ and } t'=\cos u'=[(x-\xi)\,l_1+(y-\eta)\,l_2+(z-\zeta)\,l_3]/\rho' L \quad \ldots(5.16)$$

which gives as before

$$1/r'^3 = L^{-3}\,[1+3L^{-2}\{(x-\xi)\,l_1+(y-\eta)\,l_2+(z-\zeta)\,l_3\}$$
$$+\tfrac{3}{2}L^{-2}\{5L^{-2}[(x-\xi)\,l_1+(y-\eta)\,l_2+(z-\zeta)\,l_3]^2$$
$$-[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]\}\ldots]. \quad \ldots(5.17)$$

Substituting it in the second term on the right of (5.4), it becomes

$$\Sigma' Sa^2r^{-1}[r^2 - (x\xi + y\eta + z\zeta) - (l_1x + l_2y + l_3z)].$$
$$\times L^{-3}[1 + 3L^{-2}\{(xl_1 + yl_2 + zl_3) - (\xi l_1 + \eta l_2 + \zeta l_3)\}$$
$$+ \tfrac{3}{2}L^{-2}[5L^{-2}\{(xl_1 + yl_2 + zl_3) - (\xi l_1 + \eta l_2 + \zeta l_3)\}^2$$
$$- \{(r^2 + a^2) - 2(x\xi + y\eta + z\zeta)\}]\ldots]. \quad \ldots(5.18)$$

Multiplying term by term and effecting summation our S, using results (4.5) and putting $r = a$, we get the above as

$$= \sum{}' a^3 \left[\frac{24}{L^5} z\zeta - \frac{90z\zeta l_3^2}{L^7}\right] - \sum{}' \frac{8a}{L^3} z\zeta - \frac{24a^3}{L^5} z\zeta$$
$$- \sum{}' \left[\frac{3a}{L^5}(-8z\zeta l_3^2) + \frac{3a}{2L^5} S\left\{\frac{5}{L^2}(xl_1 + yl_2 + zl_3)^3\right.\right.$$
$$+ (xl_1 + yl_2 + zl_3)\cdot(\xi l_1 + \eta l_2 + \zeta l_3)^2 - 2(xl_1 + yl_2 + zl_3)^2$$
$$\left.\left.\times (\xi l_1 + \eta l_2 + \zeta l_3)\right\} - \{2a^2(0) - 2(8z\zeta)\}\ldots\right]. \quad \ldots(5.19)$$

Simplifying the above by using the results

$$S(xl_1 + yl_2 + zl_3)^2(x\xi + y\eta + z\zeta) = (x^2l_1^2 + y^2l_2^2 + z^2l_3^2)\, 8z\zeta$$

etc. and collecting the various terms , we get from (5.4) and (5.19)

$$\sum{}' \frac{-8a}{L^3} z\zeta + \frac{24a^3}{L^5} z\zeta + \frac{24a^3}{L^5} z\zeta + \frac{24a}{L^5} z\zeta l_3^2 - \frac{120a^3}{L^7} z\zeta l_3^2$$
$$- \frac{60az\zeta}{L^7}\{(x^2l_1^2 + y^2l_2^2 + z^2l_3^2) + (\xi^2l_1^2 + \eta^2l_2^2 + \zeta^2l_3^2)\}$$
$$+ \sum{}' \frac{8az\zeta}{L^3} - \sum{}' \frac{24az\zeta l_3^2}{L^5}. \quad \ldots(5.20)$$

Since

$$\sum{}' \frac{l_1^2}{L^7} = \sum{}' \frac{l_2^2}{L^7} = \sum{}' \frac{l_3^2}{L^7} = \tfrac{1}{3}\sum{}' \frac{l_1^2 + l_2^2 + l_3^2}{L^7}$$
$$= \tfrac{1}{3}\sum{}' \frac{1}{L^5}$$

we have

$$\sum{}' (48 - 40)\frac{a^3z\zeta}{L^5} - \frac{60az\zeta}{L^7}\{(x^2l_1^2 + y^2l_2^2 + z^2l_3^2)$$
$$+ (\xi^2l_1^2 + \eta^2l_2^2 + \zeta^2l_3^2\}$$

or

$$\sum{}' \frac{8a^3z\zeta}{L^5} - \frac{60az\zeta}{L^7}\{(x^2l_1^2 + y^2l_2^2 + z^2l_3^2) + (\xi^2l_1^2 + \eta^2l_2^2 + \zeta^2l_3^2)\}. \quad \ldots(5.21)$$

Combining (5.13) and (5.21) we get a polynomial expansion for the kernel

$$K(P,Q) = -\frac{a^2}{2\pi}\cdot H_r\Big|_{r=a}$$

$$= \frac{1}{2\pi}\Big[\frac{28z\zeta}{a^2} + 8a^3 z\zeta \sum{}' \frac{1}{L^5} - \frac{60z\zeta}{a^6}\{3(x^2\xi^2 + y^2\eta^2) + z^2\zeta^2\}$$

$$- 60az\zeta \sum{}' \frac{1}{L^7}\{(x^2 l_1^2 + y^2 l_2^2 + z^2 l_3^2)$$

$$+ (\xi^2 l_1^2 + \eta^2 l_2^2 + \zeta^2 l_3^2)\}\ldots\Big]. \qquad \ldots(5.22)$$

6. Properties of the Kernel and Solution of the Integral Equation

From (5.22) we see that the kernel K is an odd function of z and vanishes for $z = 0$. It is an even function of x and y such that $\frac{\partial K}{\partial x}\Big|_{x=0} = \frac{\partial K}{\partial y}\Big|_{y=0} = 0.$

If P and Q, on the sphere, have spherical polar co-ordinate variables (a, θ, φ) and (a, θ', φ') respectively, then the integral equation (2.13) for the density μ becomes

$$\mu(\theta, \varphi) = \lambda \int_0^{\pi/2}\int_0^{\pi/2} K(\theta, \varphi; \theta', \varphi')\, \mu(\theta', \varphi') \sin\theta' d\theta' d\varphi' + \lambda\alpha \cos\theta \quad \ldots(6.1)$$

where

$$K(\theta, \varphi, \theta', \varphi') = \frac{1}{2\pi}\Big\{\Big(28 + 8a^5 \sum{}' \frac{1}{L^5}\Big)\cos\theta\cos\theta'$$

$$- 60\cos\theta\cos\theta'\{3(\sin^2\theta\cos^2\varphi\,.\,\sin^2\theta'\cos^2\varphi'$$

$$+ \sin^2\theta\sin^2\varphi\sin^2\theta'\sin^2\varphi') + \cos^2\theta\cos^2\theta'\}$$

$$- 60a^7\cos\theta\cos\theta' \sum{}' \frac{1}{L^7}\{(\sin^2\theta\cos^2\varphi l_1^2$$

$$+ \sin^2\theta\sin^2\varphi l_2^2 + \cos^2\theta l_3^2) + (\sin^2\theta'\cos^2\varphi' l_1^2$$

$$+ \sin^2\theta'\sin^2\varphi' l_2^2 + \cos^2\theta' l_3^2))\}. \qquad \ldots(6.2)$$

The singularities of the kernel have been removed by expansion in terms of polynomials. Taking the first approximation up to the second degree terms and considering μ to be independent of 'φ' the integral equation reduces to

$$\mu(\theta) = \frac{\lambda\pi}{2}\int_0^{\pi/2} K_1(\theta, \theta')\, \mu(\theta') \sin\theta' d\theta' + \lambda\alpha\cos\theta \qquad \ldots(6.3)$$

where

$$K_1(\theta, \theta') = \frac{1}{2\pi}\Big(28 + 8a^5 \sum{}' \frac{1}{L^5}\Big)\cos\theta\cos\theta'. \qquad \ldots(6.4)$$

Similarly, higher approximations of the kernel K_2(up to 4th degree) and K_3(up to 6th degree) may be picked up from (6.2) and the integral equation (6.1) solved recurrsively. The standard method of solution of integral equation (6.3) is to reduce it to a system of simultaneous algebraic equations for nodal values of $\mu(\theta, \varphi)$ by approximating the integral. For representative calculations for θ and ϕ for 15°(15°)75°, we have the system reduced to

$$\mu_i = A_{ij}\,\mu_j + B_i, \quad i, j = 1, 2, 3, \ldots, 25. \qquad \ldots(6.5)$$

The system is solved by the diagonalization process. If B_i denote the values of $\lambda \cos\theta$ for various locations, then (6.5) will yield μ_i/α at them.

The potential at any point P then would be given by (2.7) employing the generalized Green's function $H(P, Q)$ as given by (4.9).

An identical expansion and summation process is carried out, a polynomial approximant of (2.7) would give the desired temperature distribution. The following polynomial expansions are employed for the two regions R and R'. For exterior region to the sphere $r > a$

$$H(P, Q) = \frac{8z\zeta}{r^3} - \frac{12a^2z\zeta}{r^5} + \frac{20z\zeta}{r^7}\{3(x^2\xi^2 + y^2\eta^2) + z^2\zeta^2\} \qquad \ldots(6.6)$$

and for the interior region $r < a$,

$$H(P, Q) = \frac{8z\zeta}{a^3} - \frac{12r^2z\zeta}{a^5} + \frac{20z\zeta}{a^7}\{3(x^2\xi^2 + y^2\eta^2) + z^2\zeta^2\} \qquad \ldots(6.7)$$

For P on C,

$$H(P, Q) = -\frac{4z\zeta}{a^3} + \frac{20z\zeta}{a^7}\{3(x^2\xi^2 + y^2\eta^2) + z^2\zeta^2\} \qquad \ldots(6.8)$$

to which both (6.6) and (6.7) reduce for $r = a$.

Thus temperature distribution inside the cube is calculated numerically at nodal points. It is observed that the temperature exhibits the following properties:

(i) it decreases with increase of θ

(ii) it is independent of φ, i.e. the same in horizontal planes

(iii) it approaches the mean $\frac{1}{2}(T_1 + T_2)$ near the xy plane

(iv) it increases as the insulated planes are approached

(v) The temperature gradient decreases as the temperature differential $T_1 \sim T_2$ increases (curves become more flat).

Numerical verification of the boundary conditions and Plemelj formulae (Kellogg 1929, p. 164) can also be carried out. For this, we need to compare the values of σ_1/σ_2 with the ratio

$$\frac{\partial U}{\partial n_-} \Big/ \frac{\partial U}{\partial n_+} = \left(\cos\theta + \frac{\partial V}{\partial n_-}\right) \Big/ \left(\cos\theta + \frac{\partial V}{\partial n_+}\right). \qquad \ldots(6.9)$$

The values of $\frac{\partial V}{\partial n_-}$ and $\frac{\partial V}{\partial n_+}$ can be evaluated numerically by the method of analytic continuation and formula

$$\tfrac{1}{2}\left(\frac{\partial V}{\partial n_-} - \frac{\partial V}{\partial n_+}\right) = \mu(\theta, \varphi) \qquad \ldots(6.10)$$

due to Plemelj, verified.

It was observed that the first approximation K_1 to the kernel containing terms up to the second degree as given by (6.4) and the mesh net of 5×5, yielding the system of simultaneous equations (6.5) gave numerical accuracy up to the third place of decimal in the verification of (6.9) and (6.10). Finally temperature distribution graphs may be plotted.

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